

ε -STRONG SIMULATION FOR MULTIDIMENSIONAL STOCHASTIC DIFFERENTIAL EQUATIONS VIA ROUGH PATH ANALYSIS

J. BLANCHET, X. CHEN, AND J. DONG.

ABSTRACT. Consider a multidimensional diffusion process $X = \{X(t) : t \in [0, 1]\}$. Let $\varepsilon > 0$ be a *deterministic*, user defined, tolerance error parameter. Under standard regularity conditions on the drift and diffusion coefficients of X , we construct a probability space, supporting both X and an explicit, piecewise constant, fully simulatable process X_ε such that

$$\sup_{0 \leq t \leq 1} \|X_\varepsilon(t) - X(t)\|_\infty < \varepsilon$$

with probability one. Moreover, the user can adaptively choose $\varepsilon' \in (0, \varepsilon)$ so that $X_{\varepsilon'}$ (also piecewise constant and fully simulatable) can be constructed conditional on X_ε to ensure an error smaller than $\varepsilon' > 0$ with probability one. Our construction requires a detailed study of continuity estimates of the Ito map using Lyon's theory of rough paths. We approximate the underlying Brownian motion, jointly with the Lévy areas with a deterministic ε error in the underlying rough path metric.

1. INTRODUCTION

Consider the Ito Stochastic Differential Equation (SDE)

$$(1) \quad dX(t) = \mu(X(t))dt + \sigma(X(t))dZ(t), \quad X(0) = x(0)$$

where $Z(\cdot)$ is a d' -dimensional Brownian motion, and $\mu(\cdot) : R^d \rightarrow R^d$ and $\sigma(\cdot) : R^d \rightarrow R^{d \times d'}$ satisfy suitable regularity conditions. We shall assume, in particular, that both $\mu(\cdot)$ and $\sigma(\cdot)$ are Lipschitz continuous so that a strong solution to the SDE is guaranteed to exist. Additional assumptions on the first and second order derivatives of $\mu(\cdot)$ and $\sigma(\cdot)$ which are standard in the theory of rough paths will be discussed in the sequel.

Our contribution in this paper is the joint construction of $X = \{X(t) : t \in [0, 1]\}$ and a family of processes $X_\varepsilon = \{X_\varepsilon(t) : t \in [0, 1]\}$, for each $\varepsilon \in (0, 1)$, supported on a probability space (Ω, \mathcal{F}, P) , and such that the following properties hold:

- (T1) The process X_ε is piecewise constant, with finitely many discontinuities in $[0, 1]$.
- (T2) The process X_ε can be simulated exactly and, since it takes only finitely many values, its path can be fully stored.
- (T3) We have that with P -probability *one*

$$(2) \quad \sup_{t \in [0, 1]} \|X_\varepsilon(t) - X(t)\|_\infty < \varepsilon.$$

- (T4) For any $m > 1$ and $0 < \varepsilon_m < \dots < \varepsilon_1 < 1$ we can simulate X_{ε_m} conditional on $X_{\varepsilon_1}, \dots, X_{\varepsilon_{m-1}}$.

We refer to the family of procedures that achieve the construction of such family $\{X_\varepsilon : \varepsilon \in (0, 1)\}$ as *Tolerance-Enforced Simulation* (TES) or ε -strong simulation methods.

Let us discuss some considerations that motivate our study. We first discuss how this paper relates to the current literature on ε -strong simulation of stochastic processes which is a recent area of research. The paper of [6] provides the construction of X_ε satisfying only (T1) to (T3). In particular, bound (2) is satisfied for a given fixed $\varepsilon_0 = \varepsilon > 0$ and it is not clear how to jointly simulate $\{X_{\varepsilon_m}\}_{m \geq 1}$ as

$\varepsilon_m \searrow 0$ applying the technique in [6]. The motivation of constructing X_{ε_0} for [6] came from the desire to produce exact samples from a one dimensional diffusion $X(\cdot)$ satisfying (1) but assuming $\sigma(\cdot)$ constant. The authors in [6] were interested in extending the applicability of an algorithm introduced by Beskos and Roberts, see [2]. The procedure of Beskos and Roberts, applicable only to one dimensional diffusions, imposed strong boundedness assumptions on the drift coefficient and its derivative. The technique in [6] enabled the extension by using a localization technique that allowed to apply the algorithm in [2] in great generality; see also [3] for another extension.

The assumption of a constant diffusion coefficient comes at basically no cost in generality in the context of one dimensional diffusions because one can always apply Lamperti's (one-to-one) transformation in order to recast the simulation problem to one involving a diffusion with constant $\sigma(\cdot)$. Such transformation cannot be generally applied in higher dimensions.

The paper of [4] extends the work of [2] in that their algorithms satisfy (T1) to (T4). The paper [10] not only provides an additional extension which allows to deal with one dimensional SDEs with jumps, but also contains a comprehensive discussion on exact and ε -strong simulation for SDEs. Property (T4) in the definition of TES is desirable because it provides another approach at constructing unbiased estimators for expectations of the form $Ef(X)$, where $f(\cdot)$ is, say, a continuous function of the sample path X . In order to see this, let us assume for simplicity that $f(\cdot)$ is positive and Lipschitz continuous in the uniform norm with Lipschitz constant K . Then, let T be any positive random variable with a strictly positive density $g(\cdot)$ on $[0, \infty)$ and define

$$(3) \quad Z := I(f(X) > T) / g(T).$$

Observe that

$$EZ = E(E(Z|X)) = E \int_0^\infty I(f(X) > t) \frac{g(t)}{g(t)} dt = Ef(X),$$

so EZ is an unbiased estimator for $Ef(X)$. So, if Properties (1) to (4) can hold, it is possible to simulate Z by noting that $f(X_\varepsilon) > T + K\varepsilon$ implies $f(X) > T$ and if $f(X_\varepsilon) < T - K\varepsilon$, then $f(X) \leq T$. Since (T4) allows to keep simulating as ε becomes smaller and T is independent of X_ε with a positive density $g(\cdot)$, then one eventually is able to simulate Z exactly.

The major obstacle involved in developing exact sampling algorithms for multidimensional diffusions is the fact that $\sigma(\cdot)$ cannot be assumed to be constant. Moreover, even in the case of multidimensional diffusions with constant $\sigma(\cdot)$, the one dimensional algorithms developed so far can only be extended to the case in which the drift coefficient $\mu(\cdot)$ is the gradient of some function, that is, if $\mu(x) = \nabla v(x)$ for some $v(\cdot)$. The reason is that in this case one can represent the likelihood ratio $L(t)$, between the solution to (1) and Brownian motion (assuming $\sigma = I$ for simplicity) involving a Riemann integral as follows

$$L(t) = \exp \left(\int_0^t \mu(X(s)) dX(s) - \frac{1}{2} \int_0^t \|\mu(X(s))\|_2^2 ds \right) = \frac{\exp(v(X(t)))}{\exp(v(X(0)))} \exp \left(-\frac{1}{2} \int_0^t \lambda(X(s)) ds \right),$$

for $\lambda(x) = \Delta v(x) + \|\nabla v(x)\|_2^2$. The fact that the stochastic integral can be transformed into a Riemann integral facilitates the execution of acceptance rejection because one can interpret (up to a constant and using localization as in [6]) the exponential of the integral of $\lambda(\cdot)$ as the probability that no arrivals occur in a Poisson process with a stochastic intensity. Such event (i.e. no arrivals) can be simulated by thinning.

So, our motivation in this paper is to investigate a novel approach that allows to study ε -strong simulation for multidimensional diffusions in substantial generality, without imposing the assumption that $\sigma(\cdot)$ is constant or that a Lamperti-type transformation can be applied. Given the previous discussion on the connections between exact sampling and ε -strong simulation, and the limitations of the current techniques, we believe that our results here provide an important step in the development of exact sampling algorithms for general multidimensional diffusions. We plan to report

on these implications in future papers. Our results already allow to obtain unbiased samples of expectations involving sample path functionals of Brownian motion via the estimator (3). However, as noted in [4] that the expected number of random variables required to simulate Z is typically infinite. The recent paper [10] discusses via numerical examples the practical limitations of these types of estimators. The work of [11], which also proposes unbiased estimators for the expectation of Lipschitz continuous functions of $X(1)$ using randomized multilevel Monte Carlo, also with infinite expected termination time except for cases when Lévy can be simulated exactly. The authors in [1] also use rough path analysis for Monte Carlo estimation, but their focus is on connections to multilevel techniques and not on ε -strong simulation.

In this paper we concentrate only on what *is possible to do* in terms of ε -strong simulation procedures and how to enable the use of rough path theory for ε -strong simulation. Other research avenues that we plan to investigate and which leverage off our development in this paper involves quantification of model uncertainty using the fact that our ε -strong simulation algorithms in the end are uniform among a large class of drift and diffusion coefficients.

Finally, we note that in order to build our Tolerance-Enforced Simulation procedure we had to obtain new tools for the analysis of Lévy areas and associated conditional large deviations results given the increments of Brownian motion. We believe that these results might be of independent interest.

The rest of the paper is organized as follows. In Section 2 we describe the two main results of the paper. The first of them, Theorem 1, provides an error bound between the solution to the SDE described in (1) and a suitable piecewise constant approximation. The second result, Theorem 2, refers to the procedures that are involved in simulating the bounds, jointly with the piecewise constant approximation, thereby yielding (2). Section 3 is divided into three subsections and it builds the elements behind the proof of Theorem 2. As it turns out, one needs to simulate bounds on the so-called Hölder norms of the underlying Brownian motion and the corresponding Lévy areas. So, Section 3 first studies some basic estimates of for Brownian motion obtained out of its wavelet synthesis. Section 4 is also divided in several parts, corresponding to the elements of rough path theory required to analyze the SDE described in (1) as a continuous map of Brownian motion under a suitable metric (described in Section 2). While the final form of the estimates in Section 4 might be somewhat different than those obtained in the literature on rough path analysis, the techniques that we use there are certainly standard in that literature. We have chosen to present the details because the techniques might not be well known to the Monte Carlo simulation community and also because our emphasis is in finding explicit constants (i.e. bounds) that are amenable to simulation.

2. MAIN RESULTS

Our approach consists in studying the process X as a transformation of the underlying Brownian motion Z . Such transformation is known as the Ito-Lyons map and its continuity properties are studied in the theory of rough paths, pioneered by T. Lyons, in [9]. A rough path is a trajectory of unbounded variation. The theory of rough paths allows to define the solution to an SDE such as (1) in a path-by-path basis (free of probability) by imposing constraints on the regularity of the iterated integrals of the underlying process Z . Namely, integrals of the form

$$(4) \quad A_{i,j}(s, t) = \int_s^t (Z_i(u) - Z_i(s)) dZ_j(u).$$

The theory results in different interpretations of the solution to (1) depending on how the iterated integrals of Z are interpreted. In this paper, we interpret the integral in (4) in the sense of Ito.

It turns out that the Ito-Lyons map is continuous under a suitable α -Hölder metric defined in the space of rough paths. In particular, such metric can be expressed as the maximum of the following

two quantities:

$$(5) \quad \|Z\|_\alpha := \sup_{0 \leq s < t \leq 1} \frac{\|Z(t) - Z(s)\|_\infty}{|t - s|^\alpha},$$

$$(6) \quad \|A\|_{2\alpha} := \sup_{0 \leq s < t \leq 1} \max_{1 \leq i, j \leq d'} \frac{|A_{i,j}(s, t)|}{|t - s|^{2\alpha}}.$$

In the case of Brownian motion, as we consider here, we have that $\alpha \in (1/3, 1/2)$. It is shown in [7], that under suitable regularity conditions on $\mu(\cdot)$ and $\sigma(\cdot)$, which we shall discuss momentarily, the Euler scheme provides an almost sure approximation in uniform norm to the solution to the SDE (1). Our first result provides an explicit characterization of all of the (path-dependent) quantities that are involved in the final error analysis (such as $\|Z\|_\alpha$ and $\|A\|_{2\alpha}$), the difference between our analysis and what has been done in previous developments is that ultimately we must be able to implement the Euler scheme jointly with the path-dependent quantities that are involved in the error analysis. So, it is not sufficient to argue that there exists a path-dependent constant that serves as a bound of some sort, we actually must provide a suitable representation that can be simulated in finite time.

In order to provide our first result, we introduce some notations. Let D_n denote the dyadic discretization of order n and Δ_n denote the mesh of the discretization. Specifically, $D_n := \{t_0^n, t_1^n, \dots, t_{2^n}^n\}$ where $t_k^n = k/2^n$ for $k = 0, 1, 2, \dots, 2^n$ and $\Delta_n = 1/2^n$. Suppose we have a discretized approximation scheme.

Given $\hat{X}^n(0) = x(0)$, define $\{\hat{X}^n(t) : t \in D_n\}$ by the following recursion:

$$(7) \quad \hat{X}_i^n(t_{k+1}^n) = \hat{X}_i^n(t_k^n) + \mu_i(\hat{X}^n(t_k^n))\Delta_n + \sigma_i(\hat{X}^n(t_k^n))(Z(t_{k+1}^n) - Z(t_k^n)),$$

and let $\hat{X}^n(t) = \hat{X}^n(\lfloor t \rfloor)$ where $\lfloor t \rfloor = \max\{t_k^n : t_k^n \leq t\}$ for $t \in [0, 1]$. We denote

$$R_{i,j}^n(t_l^n, t_m^n) := \sum_{k=l+1}^m A_{i,j}(t_{k-1}^n, t_k^n).$$

and for fixed $\beta \in (1 - \alpha, 2\alpha)$, write

$$\Gamma_R := \sup_n \sup_{0 \leq s < t \leq 1, s, t \in D_n} \max_{1 \leq i, j \leq d'} \frac{|R_{i,j}^n(s, t)|}{|t - s|^\beta \Delta_n^{2\alpha - \beta}}.$$

We also redefine $\|Z\|_\alpha$ and $\|A\|_{2\alpha}$ as

$$\begin{aligned} \|Z\|_\alpha &:= \sup_n \sup_{0 \leq s < t \leq 1, s, t \in D_n} \frac{\|Z(t) - Z(s)\|_\infty}{|t - s|^\alpha}, \\ \|A\|_{2\alpha} &:= \sup_n \sup_{0 \leq s < t \leq 1, s, t \in D_n} \max_{1 \leq i, j \leq d'} \frac{|A_{i,j}(s, t)|}{|t - s|^{2\alpha}}. \end{aligned}$$

The new definitions are equivalent to (5) and (6) since both Z and A are continuous processes.

Theorem 1. *Suppose that there exists a constant M such that $\|\mu\|_\infty \leq M$, $\|\mu'\|_\infty \leq M$ and $\|\sigma^{(i)}\|_\infty \leq M$ for $i = 0, 1, 2, 3$. Then it is well known that a solution to X can be constructed path-by-path (see [7] and Section 4). Moreover, if $\|Z\|_\alpha \leq K_\alpha < \infty$, $\|A\|_{2\alpha} \leq K_{2\alpha} < \infty$, and $\Gamma_R < K_R$, we can compute G explicitly in terms of M , K_α , $K_{2\alpha}$ and K_R , such that*

$$\sup_{t \in [0, 1]} \|\hat{X}^n(t) - X(t)\|_\infty \leq G \Delta_n^{2\alpha - \beta}.$$

Remark: A recipe that explains step-by-step how to compute G is given in the appendix to this section.

Using Theorem 1, we can proceed to state the main contribution of this paper.

Theorem 2. *In the context of here Theorem 1, there is an explicit Monte Carlo procedure that allows to simulate random variables K_α , $K_{2\alpha}$ and K_R jointly with $\{Z(t) : t \in D_n\}$ for any $n \geq 1$. In turn, given any deterministic $\varepsilon > 0$ we can select N_0 sufficiently large, such that for $n \geq N_0$*

$$(8) \quad \sup_{t \in [0,1]} \|\hat{X}^n(t) - X(t)\|_\infty \leq \varepsilon,$$

with probability one. Moreover, conditional on $\hat{X}^n(\cdot)$ we can subsequently refine our approximation to produce $\hat{X}^{n'}(\cdot)$ with the property that $\sup_{t \in [0,1]} \|\hat{X}^{n'}(t) - X(t)\|_\infty \leq \varepsilon'$ for any $\varepsilon' < \varepsilon$.

Remark: An explicit description of the algorithm involved in the Monte Carlo procedure of Theorem 2 is given in Algorithm II at the end of Section 3.6 and the discussion that follows it. The discussion in the remark that follows Algorithm II explains how to further refine the discretization to obtain $\hat{X}^{n'}(\cdot)$.

2.1. On Relaxing Boundedness Assumptions. The construction of $\hat{X}^n(\cdot)$ in order to satisfy (8) assumes that $\|\mu\|_\infty \leq M$, $\|\mu^{(1)}\|_\infty \leq M$ and $\|\sigma^{(i)}\|_\infty \leq M$ for $i = 0, 1, 2, 3$. While these assumptions are strong we can relax them. In particular, as we shall argue now. Theorem 2 extends directly to the case in which μ and σ are Lipschitz continuous, with μ differentiable and σ is three times differentiable. Since μ and σ are Lipschitz continuous we know that $X(\cdot)$ has a strong solution which is non-explosive.

We can always construct μ_M and σ_M so that $\mu^{(i)}(x) = \mu_M^{(i)}(x)$ for $\|x\|_\infty \leq c_M$ and $i = 0, 1$, and $\sigma^{(i)}(x) = \sigma_M^{(i)}(x)$ for $\|x\|_\infty \leq c_M$ for $i = 0, 1, 2, 3$. Also we can choose $c_M \rightarrow \infty$, and $\|\mu_M\|_\infty \leq M$, $\|\mu_M^{(1)}\|_\infty \leq M$ and $\|\sigma_M^{(i)}\|_\infty \leq M$ for $i = 0, 1, 2, 3$.

For $M \geq 1$ we consider the SDE (1) with μ_M and σ_M as drift and diffusion coefficients, respectively, and let $X_M(\cdot)$ be the corresponding solution to (1). We start by picking some $M_0 \geq 1$ such that $\varepsilon < c_{M_0}$ and let $M = M_0$. Then run Algorithm II to produce $\{\hat{X}_M^n(t) : t \in [0, 1]\}$, which according to Theorem 2 satisfies,

$$\sup_{t \in [0,1]} \|\hat{X}_M^n(t) - X_M(t)\|_\infty \leq \varepsilon.$$

Note that only Steps 5 to 8 in Algorithm II depend on the SDE (1), through the evaluation of G , which depends on M and so we write $G_M := G$. If $\sup_{t \in [0,1]} \|\hat{X}_M^n(t)\|_\infty \leq c_M - \varepsilon$, then we must have that $X(t) = X_M(t)$ for $t \in [0, 1]$ and we are done. Otherwise, we let $M \leftarrow 2M$ and run again only Steps 5 to 8 of Algorithm II. We repeat doubling M and re-running Steps 5 to 8 (updating G_M) until we obtain a solution for which $\sup_{t \in [0,1]} \|\hat{X}_M^n(t)\|_\infty \leq c_M - \varepsilon$. Eventually this must occur because

$$\lim_{M \rightarrow \infty} \sup_{t \in [0,1]} \|X_M(t) - X(t)\|_\infty = 0$$

almost surely and $X(\cdot)$ is non explosive.

2.2. Appendix to Theorem 1: The Evaluation of G . We next summarize the way to calculate G in terms of M , K_α , $K_{2\alpha}$ and K_R .

Procedure A.

(1) Find δ and $C_i(\delta)$ for $i = 1, 2, 3$ that satisfies the following relations:

$$C_1(\delta) \geq C_3(\delta)\delta^{2\alpha} + M\delta^{1-\alpha} + dM\|Z\|_\alpha + d^3M^2\|A\|_{2\alpha}\delta^\alpha$$

$$C_2(\delta) \geq C_3(\delta)\delta^\alpha + d^3M^2\|A\|_{2\alpha}$$

$$C_3(\delta) \geq \frac{2}{1-2^{1-3\alpha}} (MC_1(\delta) + dMC_1(\delta)^2\|Z\|_\alpha + d^2MC_2(\delta)\|Z\|_\alpha + 2d^3M^2C_1(\delta)\|A\|_\alpha)$$

(2) Set $C_1 = \frac{2}{\delta}C_1(\delta)$, $C_2 = \frac{2}{\delta}(C_2(\delta) + MC_1 + dMC_1\|Z\|_\alpha)$ and

$$C_3 = \frac{2}{1-2^{1-3\alpha}}(MC_1 + dMC_1^2\|Z\|_\alpha + d^2MC_2\|Z\|_\alpha + 2d^3M^2C_1\|A\|_\alpha)$$

(3) Find δ' and $B_i(\delta')$ for $i = 1, 2, 3$ that satisfies the following relations:

$$B_1(\delta') > B_3(\delta')\delta'^{2\alpha} + 2M\delta'^{1-\alpha} + 2M\|Z\|_\alpha + 4M^2\|A\|_{2\alpha}\delta'^\alpha$$

$$B_2(\delta') > B_3(\delta')\delta'^\alpha + 4M^2\|A\|_{2\alpha}$$

$$B_3(\delta') > \frac{4}{1-2^{1-3\alpha}} (MB_1(\delta') + MB_1(\delta')^2\|Z\|_\alpha + MB_2(\delta')\|Z\|_\alpha + 2M^2B_1(\delta')\|A\|_\alpha)$$

(4) Set $B = \frac{2}{\delta'}B_1(\delta')$

(5) Set $G_1 = (1 + B)C_3$

(6) Find δ'' and $C_4(\delta'')$ such that

$$B\delta''^\alpha \leq 2^{\alpha+\beta} - 2$$

$$C_4(\delta'') \geq 2(1 - \frac{2 + B\delta''^\alpha}{2^{\alpha+\beta}})^{-1}(Bd^3M^2\Gamma_R + 2d^3M^2C_1\Gamma_R)$$

(7) Set $C_4 = (1 + B\delta''^\alpha)C_4(\delta''^3M^2\Gamma_R + 2d^3M^2C_1\Gamma_R)/\delta''$

(8) Set $G_2 = C_4 + d^3M^2\Gamma_R$

(9) Set $G = G_1 + G_2$

3. USE OF WAVELETS TO BOUND α -HÖLDER NORMS, TOLERANCE-ENFORCED SIMULATION, AND THE PROOF OF THEOREM 2.

Our goal in this section is to simulate the upper bounds for $\|Z\|_\alpha$, $\|A\|_{2\alpha}$ and Γ_R respectively. We will first recall Lévy-Ciesielski's construction of Brownian motion and provide a high level picture of the approach that we will follow based on "record breakers".

3.1. Wavelet Synthesis of Brownian Motion and Record Breakers . We do so by implementing the Lévy-Ciesielski construction of Brownian motion which is explained next following Steele [12] pages 34-39. First we need to define a step function $H(\cdot)$ on $[0, 1]$ by

$$H(t) = I(0 \leq t < 1/2) - I(1/2 \leq t \leq 1).$$

Then define a family of functions

$$H_k^n(t) = 2^{n/2}H(2^n t - k)$$

for all $n \geq 0$ and $0 \leq k < 2^n$. Set $H_0^0(t) = 1$ and then one obtains the following.

Theorem 3 (Lévy-Ciesielski Construction). *If $\{W_k^n : 0 \leq k < 2^n, n \geq 0\}$ is a sequence of independent standard normal random variables, then the series defined by*

$$(9) \quad Z(t) = \sum_{n=0}^{\infty} \sum_{k=0}^{2^n-1} \left(W_k^n \int_0^t H_k^n(s) ds \right)$$

converges uniformly on $[0, 1]$ with probability one. Moreover, the process $\{Z(t) : t \in [0, 1]\}$ is a standard Brownian motion on $[0, 1]$.

Eventually we will simulate the series up to a random but finite level N which can be viewed as the order of dyadic discretization. The level N is selected so that the contribution of the remaining terms (terms beyond level N) can be guaranteed to be bounded by a user defined tolerance error. We think of simulating the discretization levels sequentially, so we often refer to “time” when discussing levels.

Once we have simulated up to time (or level) N , we further decompose the analysis into two parts. One dealing with finding the upper bound for $\|Z\|_\alpha$ (Section 3.2), the other dealing with finding the upper bound for Γ_R (Section 3.3). We then combine these two parts to obtain an upper bound for $\|A\|_{2\alpha}$.

For both parts, we use a strategy based on a suitably defined sequence of “record breakers”. We ask a “yes or no” question to the future (i.e. to higher order discretization levels). The question, which corresponds to the simulation of a Bernoulli random variable, is “will there be a new record breaker?” The definition of record breakers need to satisfy the following two conditions.

Conditions:

- (1) The following event happens with probability one: beyond some random but finite time, there will be no more record breakers.
- (2) By knowing that there are no more record breakers, the contribution of the terms that we have not simulated yet are well under control (i.e. bounded by a user defined tolerance error).

Now we explain for each part, how the above strategy is applied. We have d' independent Brownian motions and we will use $W_{i,k}^n$ for $i \in \{1, \dots, d'\}$ to denote the (n, k) coefficient in the expansion (9) for the i -th Brownian motion.

For $\|Z\|_\alpha$ we say a record is broken at (i, n, k) , for $1 \leq i \leq d'$, $n \geq 0$ and $0 \leq k < 2^n$, if

$$|W_{i,k}^n| > 4\sqrt{n+1}.$$

Let $N_1 := \max\{n \geq 0 : |W_{i,k}^n| > 4\sqrt{n+1} \text{ for some } 0 \leq k < 2^n\}$. Lemma 2 shows that $E[N_1] < \infty$. Thus Condition 1 for “record breaker” is satisfied. We then check Condition 2. By Lemma 1, we have $\|Z\|_\alpha \leq 2^{2\alpha} \sum_{n=0}^{\infty} 2^{-n(1/2-\alpha)} V^n$ where $V^n = \max_{0 \leq k < 2^n} |W_k^n|$. Once we found N_1 , we have

$$\begin{aligned} \|Z\|_\alpha &\leq \sum_{n=1}^{\lceil \log_2 N_1 \rceil} 2^{-n(1/2-\alpha)} V^n + \sum_{\lceil \log_2 N_1 \rceil + 1}^{\infty} 2^{-n(1/2-\alpha)} \sqrt{n+1} \\ &\leq \sum_{n=1}^{\lceil \log_2 N_1 \rceil} 2^{-n(1/2-\alpha)} V^n + \frac{(\lceil \log_2 N_1 \rceil + 1)^{-1/2(1/2-\alpha)}}{1 - 2^{-1/2(1/2-\alpha)}}. \end{aligned}$$

For Γ_R , we first define a sequence of random walks

$$\begin{aligned} L_{i,j}^n(0) &:= 0, \\ L_{i,j}^n(k) &:= L_{i,j}^n(k-1) + (Z_i(t_{2k-1}^n) - Z_i(t_{2k-2}^n)) (Z_j(t_{2k}^n) - Z_j(t_{2k-1}^n)) \text{ for } k = 1, 2, \dots, 2^{n-1}. \end{aligned}$$

We then say a record is broken at (n, k, k') , for $n \geq 1, 0 \leq k < k' < 2^{n-1}$, if

$$|L_{i,j}^n(k') - L_{i,j}^n(k)| > (k' - k)\Delta_n^{2\alpha}.$$

Let $N_2 := \max\{n \geq 1 : |L_{i,j}^n(k') - L_{i,j}^n(k)| > (k' - k)\Delta_n^{2\alpha} \text{ for some } 0 \leq k < k' \leq 2^{n-1}\}$. Lemma 4 proves that $N_2 < \infty$ with probability 1, which justifies Condition 1. Once we found N_2 , by Lemma 5, we have

$$\Gamma_R \leq \frac{2^{-(2\alpha-\beta)}}{1 - 2^{-(2\alpha-\beta)}} \max_{n \leq N_2} \max_{1 \leq i,j \leq d'} \max_{0 \leq k < k' \leq 2^{n-1}} \left\{ \frac{|L_{i,j}^n(k') - L_{i,j}^n(k)|}{(k' - k)^\beta \Delta_n^{2\alpha}} \right\}.$$

Thus, Condition 2 is satisfied as well.

Once we established the bounds for $\|Z\|_\alpha$ and Γ_R , by Lemma 5, we have

$$\|A\|_{2\alpha} \leq \Gamma_R \frac{2}{1 - 2^{-2\alpha}} + \|Z\|_\alpha^2 \frac{2^{1-\alpha}}{1 - 2^{-\alpha}}.$$

In Section 3.6, we will explain how to simulate the random numbers $(N_1 \text{ and } N_2)$ jointly with the wavelet construction using the “record breaker” strategy introduced above. Specifically, we first find all the record breakers in sequence and then simulate the rest of the process conditional on the information of the record breakers.

3.2. Tolerance-Enforced Simulation of Bounds on α -Hölder Norms . In this section, we will explain how to use the wavelet synthesis to approximate a single Brownian motion, Z , in the α -Hölder norm, (5). Of course, since we have d' Brownian motion, ultimately the algorithm that we shall describe for such an approximation (see Algorithm I below) will be run d' independent times.

Let us define $V^n = \max_{0 \leq k < 2^n} |W_k^n|$. We have the following auxiliary lemma.

Lemma 1.

$$\|Z\|_\alpha \leq 2^{2\alpha} \sum_{n=0}^{\infty} 2^{-n(\frac{1}{2}-\alpha)} V^n.$$

Proof. For any interval $[t, t + \delta] \subset [0, 1]$, suppose $2^{-m+2} \leq \delta \leq 2^{-m+1}$, then there exists two level n dyadic points t_k^m and t_{k+1}^m such that $[t, t + \delta] \subset [t_k^m, t_{k+1}^m]$. Using the Lévy-Ciesielski construction, one can check that

$$|Z(t + \delta) - Z(t)| \leq \sum_{n=0}^m 2^{-m+\frac{n}{2}} V^n + \sum_{n=m+1}^{\infty} 2^{-\frac{n}{2}} V^n.$$

Since $\delta > 2^{-m+2}$, we have

$$\begin{aligned} \frac{|Z(t + \delta) - Z(t)|}{\delta^\alpha} &\leq 2^{2\alpha} \left(\sum_{n=0}^m 2^{-(1-\alpha)m+\frac{n}{2}} V^n + \sum_{n=m+1}^{\infty} 2^{-\frac{n}{2}+\alpha m} V^n \right) \\ &\leq 2^{2\alpha} \left(\sum_{n=0}^m 2^{-(1-\alpha)n+\frac{n}{2}} V^n + \sum_{n=m+1}^{\infty} 2^{-\frac{n}{2}+\alpha n} V^n \right) \\ &\leq 2^{2\alpha} \sum_{n=0}^{\infty} 2^{-n(\frac{1}{2}-\alpha)} V^n. \end{aligned}$$

As the interval $[t, t + \delta]$ is arbitrarily chosen, we obtain the result. \square

Owing to Lemma 1 we can now find a bound on $\|Z\|_\alpha$. Let

$$N_1 = \max\{n \geq 1 : |W_k^n| > 4\sqrt{n+1} \text{ for some } 1 \leq k \leq 2^n\}.$$

Lemma 2.

$$E(N_1) < \infty$$

Proof. We note that

$$E(N_1) \leq \sum_{n=1}^{\infty} \sum_{k=1}^{2^n} P(|W_k^n| > 4\sqrt{n+1}) \leq \sum_{n=1}^{\infty} 2^n \exp(-8n) < \infty.$$

□

The strategy is then to simulate N_1 jointly with the sequence $\{W_k^n\}$. It is important to note that N_1 is not a stopping time with respect to the filtration generated by $\{(W_k^n : 1 \leq k \leq 2^n) : n \geq 1\}$. Note that if N_1 is simulated jointly with $\{W_k^n\}$, then for $2^n + k \geq N_1 + 1$, $|W_k^n| \leq 4\sqrt{n+1}$ and thus we can compute

$$(10) \quad K_\alpha = \sum_{n=1}^{\lfloor \log_2 N_1 \rfloor} 2^{-n(\frac{1}{2}-\alpha)} V^n + \sum_{n=\log_2 N_1+1}^{\infty} 2^{-n(\frac{1}{2}-\alpha)} \sqrt{n+1} < \infty.$$

We call a pair (n, k) such that $|W_k^n| > 4\sqrt{n+1}$ a broken-record-pair. All pairs (both broken-record-pairs and non broken-record-pairs) can be totally ordered lexicographically. The distribution of subsequent pairs at which records are broken is not difficult to compute (because of the independence of the W_k^n 's). So, using a sequential acceptance / rejection procedure we can simulate all of the broken-record-pairs. Conditional on these pairs the distribution of the $\{(W_k^n : 1 \leq k \leq 2^n) : n \geq 1\}$ is straightforward to describe. Precisely, if (k, n) is a broken-record-pair, then W_k^n is conditioned on $|W_k^n| > 4\sqrt{n+1}$ and thus is straightforward to simulate. Similarly, if (k, n) is not a broken-record-pair, then W_k^n is conditioned on $|W_k^n| \leq 4\sqrt{n+1}$ and also can be easily simulated.

The simulation of the broken-record-pairs has been studied in [5], see Algorithm 2W. We synthesize their algorithm for our purposes next.

Algorithm I: Simulate N_1 jointly with the broken-record-pairs

Input: A positive parameter $\rho > 4$.

Output: A vector S which gives all the indices $l = 2^n + k$ such that (n, k) is a broken-record-pair.

Step 0: Initialize $R = 0$ and S to be an empty array.

Step 1: Set $U = 1$, $D = 0$. Simulate $V \sim \text{Uniform}(0, 1)$.

Step 2: While $U > V > D$, set $R \leftarrow R + 1$ and $U \leftarrow P(|W_k^n| \leq \rho\sqrt{\log R}) \times U$ and $D \leftarrow (1 - R^{1-\rho^2/2}) \times U$.

Step 3: If $V \geq U$, add R to the end of S , i.e. $S = [S, R]$, and return to Step 1.

Step 4: If $V \leq D$, $N = \max(S)$.

Step 5: Output S .

Remark: Observe that for every $l = 2^n + k \in S$, we can generate W_k^n conditional on the event $\{|W_k^n| > \rho\sqrt{\log l}\}$; for other $1 \leq l \leq N$ (i.e. $l \notin S$) generate W_k^n given $\{|W_k^n| \leq \rho\sqrt{\log l}\}$. Note that at the end of Algorithm 1 and after simulating W_k^n for $l = 2^n + k \leq N$ one can compute quantities such as K_α according to (10).

3.3. Analysis and Bounds of α -Hölder Norms of Lévy Areas. We shall start by stating the following representation of the Lévy area $A_{i,j}(t_k^n, t_{k+1}^n)$, which we believe is of independent interest.

Lemma 3.

$$A_{i,j}(t_k^n, t_{k+1}^n) = \sum_{h=n+1}^{\infty} \sum_{l=1}^{2^{h-n-1}} [Z_i(t_{2^{h-n}k+2l-1}^h) - Z_i(t_{2^{h-n}k+2l-2}^h)][Z_j(t_{2^{h-n}k+2l}^h) - Z_j(t_{2^{h-n}k+2l-1}^h)].$$

The inner summation inside the expression of $A_{i,j}(t_k^n, t_{k+1}^n)$ motivates the definition of the following family of processes $(L_{i,j}^n(k) : k = 0, 1, \dots, 2^{n-1})$, for $n \geq 1$:

$$\begin{aligned} L_{i,j}^n(0) &:= 0 \\ L_{i,j}^n(k) &:= L_{i,j}^n(k-1) + (Z_i(t_{2k-1}^n) - Z_i(t_{2k-2}^n))(Z_j(t_{2k}^n) - Z_j(t_{2k-1}^n)) \text{ for } k = 1, 2, \dots, 2^{n-1}. \end{aligned}$$

Using this definition and Lemma 3 we can succinctly write $A_{i,j}(t_k^n, t_{k+1}^n)$ as

$$(11) \quad A_{i,j}(t_k^n, t_{k+1}^n) = \sum_{h=n+1}^{\infty} (L_{i,j}^h(2^{h-n}(k+1)) - L_{i,j}^h(2^{h-n}k)).$$

Moreover, the following result allows to control the behavior of the terms in the previous infinite series.

Lemma 4. *There exists $N_2 < \infty$ such that for all $n \geq N_2$ and all $l < m < 2^{n-1}$ we have*

$$|L_{i,j}^n(m) - L_{i,j}^n(l)| \leq (m-l)^\beta \Delta_n^{2\alpha}.$$

Now, recall that

$$R_{i,j}^n(t_l^n, t_m^n) := \sum_{k=l+1}^m A_{i,j}(t_{k-1}^n, t_k^n).$$

A direct application of Lemmas 3 and 4 yields the next corollary.

Corollary 1.

$$R_{i,j}^n(t_l^n, t_m^n) = \sum_{h=n+1}^{\infty} (L_{i,j}^h(2^{h-n}m) - L_{i,j}^h(2^{h-n}l)).$$

We conclude this section with a proposition which summarizes the bounds that we will simulate.

Lemma 5. *Suppose that N_2 is chosen according to Lemma 4. We define*

$$\Gamma_L := \max_{1 \leq i,j \leq d'} \max_{n < N_2} \max_{0 \leq l < m \leq 2^{n-1}} \left\{ \frac{|L_{i,j}^n(m) - L_{i,j}^n(l)|}{(m-l)^\beta \Delta_n^{2\alpha}} \right\}.$$

Then

$$\Gamma_R \leq \frac{2^{-(2\alpha-\beta)}}{1 - 2^{-(2\alpha-\beta)}} \Gamma_L$$

and

$$\|A\|_{2\alpha} \leq \Gamma_R \frac{2}{1 - 2^{-2\alpha}} + \|Z\|_\alpha^2 \frac{2^{1-\alpha}}{1 - 2^{-\alpha}}.$$

3.4. Appendix to Section 3.3: Technical Proofs. We now provide the proofs of the results in the order in which they were presented in the previous section. We start by recalling the following algebraic property of the Lévy areas: for each $0 \leq r < s < t$

$$(12) \quad A_{i,j}(r, t) = A_{i,j}(r, s) + A_{i,j}(s, t) + (Z_i(s) - Z_i(r))(Z_j(t) - Z_j(s)).$$

Using this property and a simple use of the Borel-Cantelli lemma we can obtain the proof of Lemma 3.

Proof of Lemma 3. We use (12) repeatedly. First, note that

$$A_{i,j}(t_k^n, t_{k+1}^n) = A_{i,j}(t_{2k}^{n+1}, t_{2k+1}^{n+1}) + A_{i,j}(t_{2k+1}^{n+1}, t_{2k+2}^{n+1}) + (Z_i(t_{2k+1}^{n+1}) - Z_i(t_{2k}^{n+1}))(Z_j(t_{2k+2}^{n+1}) - Z_j(t_{2k+1}^{n+1})).$$

We continue, this time splitting $A_{i,j}(t_{2k}^{n+1}, t_{2k+1}^{n+1})$ and $A_{i,j}(t_{2k+1}^{n+1}, t_{2k+2}^{n+1})$, thereby obtaining

$$\begin{aligned} & A_{i,j}(t_k^n, t_{k+1}^n) \\ &= (Z_i(t_{2k+1}^{n+1}) - Z_i(t_{2k}^{n+1}))(Z_j(t_{2k+2}^{n+1}) - Z_j(t_{2k+1}^{n+1})) \\ &+ A_{i,j}(t_{2k}^{n+2}, t_{2k+1}^{n+2}) + A_{i,j}(t_{2k+1}^{n+2}, t_{2k+2}^{n+2}) \\ &+ (Z_i(t_{2k+1}^{n+2}) - Z_i(t_{2k}^{n+2}))(Z_j(t_{2k+2}^{n+2}) - Z_j(t_{2k+1}^{n+2})) \\ &+ A_{i,j}(t_{2k+2}^{n+2}, t_{2k+3}^{n+2}) + A_{i,j}(t_{2k+3}^{n+2}, t_{2k+4}^{n+2}) \\ &+ (Z_i(t_{2k+3}^{n+2}) - Z_i(t_{2k+2}^{n+2}))(Z_j(t_{2k+4}^{n+2}) - Z_j(t_{2k+3}^{n+2})). \end{aligned}$$

Iterating m times the previous splitting procedure we conclude that

$$(13) \quad \begin{aligned} A_{i,j}(t_k^n, t_{k+1}^n) &= \sum_{h=n+1}^m \sum_{l=1}^{2^{h-n-1}} [Z_i(t_{2^{h-n}k+2l-1}^h) - Z_i(t_{2^{h-n}k+2l-2}^h)][Z_j(t_{2^{h-n}k+2l}^h) - Z_j(t_{2^{h-n}k+2l-1}^h)] \\ &+ \sum_{l=1}^{2^{m-n-1}} A_{i,j}(t_{2^{m-n}k+2l-2}^m, t_{2^{m-n}k+2l-1}^m) + \sum_{l=1}^{2^{m-n-1}} A_{i,j}(t_{2^{m-n}k+2l-1}^m, t_{2^{m-n}k+2l}^m). \end{aligned}$$

We claim that

$$(14) \quad \sum_{l=1}^{2^{m-n-1}} A_{i,j}(t_{2^{m-n}k+2l-2}^m, t_{2^{m-n}k+2l-1}^m) + \sum_{l=1}^{2^{m-n-1}} A_{i,j}(t_{2^{m-n}k+2l-1}^m, t_{2^{m-n}k+2l}^m) \rightarrow 0$$

almost surely as $m \rightarrow \infty$. To see this note that

$$\begin{aligned} & P\left(\left|\sum_{l=1}^{2^{m-n-1}} A_{i,j}(t_{2^{m-n}k+2l-2}^m, t_{2^{m-n}k+2l-1}^m)\right| > 1/m\right) \\ &\leq m^2 \sum_{l=1}^{2^{m-n-1}} E(A_{i,j}^2(t_{2^{m-n}k+2l-2}^m, t_{2^{m-n}k+2l-1}^m)) = m^2 2^{m-n+1} E \int_0^{\Delta_m} Z_i^2(s) ds \\ &= m^2 2^{m-n} \Delta_m^2 = 2^n m^2 \Delta_m. \end{aligned}$$

Since $\sum_{m=1}^{\infty} m^2 \Delta_m < \infty$ we conclude by Borel-Cantelli's lemma that, almost surely, for m large enough

$$\left|\sum_{l=1}^{2^{m-n-1}} A_{i,j}(t_{2^{m-n}k+2l-2}^m, t_{2^{m-n}k+2l-1}^m)\right| < 1/m$$

and thus indeed we have (14) holds almost surely and therefore, from (13), sending $m \rightarrow \infty$ we obtain the conclusion of the lemma. \square

Then we present the proof of Lemma 4.

Proof of Lemma 4. Define

$$\mathcal{C}_n = \{|L_{i,j}^n(m) - L_{i,j}^n(l)| > (m-l)^\beta \Delta_n^{2\alpha} \text{ for some } 0 \leq l < m < 2^{n-1}\}.$$

We will show that the events $\{\mathcal{C}_n : n \geq 0\}$ occur finitely many times. Note that

$$(15) \quad P(\mathcal{C}_n) \leq \sum_{0 \leq l < m < 2^{n-1}} 2P\left((L_{i,j}^n(m) - L_{i,j}^n(l)) > (m-l)^\beta \Delta_n^{2\alpha}\right).$$

Also observe that for fixed m and n , $L_{i,j}^n(m)$ is the sum of m i.i.d. random variables, each of which is distributed as $(Z_i(t_1^n) - Z_i(t_0^n))(Z_j(t_2^n) - Z_j(t_1^n))$ and we easily evaluate

$$E \exp(\theta(Z_i(t_1^n) - Z_i(t_0^n))(Z_j(t_2^n) - Z_j(t_1^n))) = (1 - \theta^2 \Delta_n^2)^{-1/2}.$$

We apply Chernoff's bound concluding

$$P\left((L_{i,j}^n(m) - L_{i,j}^n(l)) > (m-l)^\beta \Delta_n^{2\alpha}\right) \leq \exp\left(-\theta(m-l)^\beta \Delta_n^{2\alpha} - \frac{1}{2}(m-l) \log(1 - \theta^2 \Delta_n^2)\right).$$

Select $\theta = \theta'(m-l)^{-1/2} \Delta_n^{-1}$ for $\theta' \in (0, 1/4)$

$$P\left((L_{i,j}^n(m) - L_{i,j}^n(l)) > (m-l)^\beta \Delta_n^{2\alpha}\right) \leq \exp\left(-\theta'(m-l)^{\beta-1/2} \Delta_n^{2\alpha-1} + 1\right).$$

Hence,

$$P(\mathcal{C}_n) \leq \sum_{0 \leq l < m \leq 2^{n-1}} 2 \exp\left(-\theta'(m-l)^{\beta-1/2} \Delta_n^{2\alpha-1} + 1\right) \leq 2^{2n} \exp\left(-\theta' 2^{n(1-2\alpha)}\right).$$

Since $2\alpha < 1$ we clearly have that

$$\sum_{n=1}^{\infty} P(\mathcal{C}_n) < \infty$$

and by Borel-Cantelli's lemma we conclude that $P(\mathcal{C}_n \text{ infinitely often}) = 0$ which in turns yields the existence of such N_2 . \square

The proof of Corollary 1 follows directly from Lemma 3 and Lemma 4.

Proof of Corollary 1. Using Lemma 3 we obtain that

$$(16) \quad R_{i,j}^n(t_l^n, t_m^n) = \sum_{k=l+1}^m \sum_{h=n+1}^{\infty} (L_{i,j}^h(2^{h-n}(k+1)) - L_{i,j}^h(2^{h-n}k)).$$

On the other hand, due to Lemma 4 if $n \geq N_2$

$$\sum_{k=l+1}^m \sum_{h=n+1}^{\infty} |L_{i,j}^h(2^{h-n}(k+1)) - L_{i,j}^h(2^{h-n}k)| \leq \sum_{k=l+1}^m \sum_{h=n+1}^{\infty} (2^{-n}(k+1) - 2^{-n}k)^\beta \Delta_h^{2\alpha-\beta} < \infty$$

because $\beta < 2\alpha$. Thus (by Fubini's theorem) the order of the summations in (16) can be exchanged and we obtain the result. \square

Finally, the last proof of the section.

Proof of Lemma 5. We start by showing the bound on Γ_R . By the definition of N_2 and Γ_L , for any n

$$|L_{i,j}^n(m) - L_{i,j}^n(l)| \leq \Gamma_L(m-l)^\beta \Delta_n^{2\alpha}.$$

Consequently, for any $0 \leq l < m \leq 2^{n-1}$,

$$\begin{aligned} |R_{i,j}^n(t_l^n, t_m^n)| &\leq \sum_{h=n+1}^{\infty} \left| L_{i,j}^h(2^{h-n}m) - L_{i,j}^h(2^{h-n}l) \right| \\ &\leq \sum_{h=n+1}^{\infty} \Gamma_L(m-l)^\beta 2^{(h-n)\beta} \Delta_h^{2\alpha} = \Gamma_L(m-l)^\beta \Delta_n^\beta \sum_{h=n+1}^{\infty} \Delta_h^{2\alpha-\beta} \\ &= \Gamma_L(t_l^n - t_m^n)^\beta \Delta_n^{2\alpha-\beta} \frac{2^{-(2\alpha-\beta)}}{1 - 2^{-(2\alpha-\beta)}}. \end{aligned}$$

Therefore, we conclude that

$$\begin{aligned} \Gamma_R &:= \max_{1 \leq i,j \leq d'} \sup_{n \geq 0} \sup_{0 \leq s < t \leq 1, s,t \in D_n} \frac{|R_{i,j}^n(s,t)|}{|t-s|^\beta \Delta_n^{2\alpha-\beta}} \\ &\leq \Gamma_L \frac{2^{-(2\alpha-\beta)}}{1 - 2^{-(2\alpha-\beta)}}. \end{aligned}$$

Let $r = \min\{h : |t_m^n - t_l^n| \geq \Delta_h\}$. For simplicity of notation, we define the following sequence of operators of time:

$$\begin{aligned} \underline{s}^h(t_l^n) &= \min\{t_k^h : t_k^h \geq t_l^n\} \\ \bar{s}^h(t_m^n) &= \max\{t_k^h : t_k^h \leq t_m^n\} \end{aligned}$$

for $r \leq h \leq n$.

Then

$$\begin{aligned} &|A_{i,j}(t_l^n, t_m^n)| \\ &\leq |A_{i,j}(t_l^n, \underline{s}^{n-1}(t_l^n))| + |A_{i,j}(\underline{s}^{n-1}(t_l^n), \bar{s}^{n-1}(t_m^n))| + |A_{i,j}(\bar{s}^{n-1}(t_m^n), t_m^n)| \\ &\quad + |Z_i(\underline{s}^{n-1}(t_l^n)) - Z_i(t_l^n)| |Z_j(\bar{s}^{n-1}(t_m^n)) - Z_j(\underline{s}^{n-1}(t_l^n))| \\ &\quad + |Z_i(\bar{s}^{n-1}(t_m^n)) - Z_i(t_m^n)| |Z_j(t_m^n) - Z_j(\bar{s}^{n-1}(t_m^n))| \end{aligned}$$

By iterating the above procedure up to level r , we have

$$\begin{aligned} &|A_{i,j}(t_l^n, t_m^n)| \\ &\leq \sum_{h=r+1}^n |A_{i,j}(\underline{s}^h(t_l^n), \underline{s}^{h-1}(t_l^n))| + |A_{i,j}(\underline{s}^r(t_l^n), \bar{s}^r(t_m^n))| + \sum_{h=r+1}^n |A_{i,j}(\bar{s}^h(t_m^n), \bar{s}^{h-1}(t_m^n))| \\ &\quad + \sum_{h=r+1}^n |Z_i(\underline{s}^h(t_l^n)) - Z_i(\underline{s}^{h-1}(t_l^n))| |Z_j(\bar{s}^{h-1}(t_m^n)) - Z_j(\underline{s}^{h-1}(t_l^n))| \\ &\quad + \sum_{h=r+1}^n |Z_i(\bar{s}^{h-1}(t_m^n)) - Z_i(\underline{s}^h(t_l^n))| |Z_j(\bar{s}^h(t_m^n)) - Z_j(\bar{s}^{h-1}(t_m^n))| \end{aligned}$$

We make the following important observations,

$$\underline{s}^{h-1}(t_l^n) - \underline{s}^h(t_l^n) = \begin{cases} 0 & \text{if } \underline{s}^{h-1}(t_l^n) = \underline{s}^h(t_l^n) \\ \Delta_h & \text{otherwise} \end{cases}$$

$$\bar{s}^h(t_m^n) - \bar{s}^{h-1}(t_m^n) = \begin{cases} 0 & \text{if } s^{h-1}(t_m^n) = \bar{s}^h(t_m^n) \\ \Delta_h & \text{otherwise} \end{cases}$$

$$\bar{s}^r(t_m^n) - \underline{s}^r(t_l^n) = \begin{cases} 0 & \text{if } \underline{s}^r(t_l^n) = \bar{s}^r(t_m^n) \\ \Delta_r & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} & \frac{|A_{i,j}(t_l^n, t_m^n)|}{(t_m^n - t_l^n)^{2\alpha}} \\ & \leq \sum_{h=r+1}^n \Gamma_R \frac{\Delta_h^{2\alpha}}{\Delta_r^{2\alpha}} + \Gamma_R + \sum_{h=r+1}^n \Gamma_R \frac{\Delta_h^{2\alpha}}{\Delta_r^{2\alpha}} + \sum_{h=r+1}^n \|Z\|_\alpha^2 \frac{\Delta_h^\alpha}{\Delta_r^\alpha} + \sum_{h=r+1}^n \|Z\|_\alpha^2 \frac{\Delta_h^\alpha}{\Delta_r^\alpha} \\ & \leq \Gamma_R \frac{2}{1 - 2^{-2\alpha}} + \|Z\|_\alpha^2 \frac{2^{1-\alpha}}{1 - 2^{-\alpha}}. \end{aligned}$$

Therefore,

$$\begin{aligned} \|A\|_{2\alpha} &:= \max_{1 \leq i \leq j \leq d'} \sup_{n \geq 1} \sup_{0 \leq s < t \leq 1; s, t \in D_n} \frac{|A_{i,j}(s)|}{|t - s|^{2\alpha}} \\ &\leq \Gamma_R \frac{2}{1 - 2^{-2\alpha}} + \|Z\|_\alpha^2 \frac{2^{1-\alpha}}{1 - 2^{-\alpha}}. \end{aligned}$$

□

3.5. Elements of Tolerance-Enforced Simulation for Bounds on α -Hölder Norms of Lévy Areas. There is some resemblance between the problem of sampling N_1 in Section 3.2, which involves a sequence of i.i.d. random variables (W_k^n) 's, and sampling of N_2 introduced in Section 3.3. However, simulation of N_2 , which is basically our main goal here, is a lot more complicated because there is fair amount of dependence on the structure of the $L_{i,j}^n(k)$'s as one varies n . Let us provide a general idea of our simulation procedure in order to set the stage for the definitions and estimates that must be studied first.

Suppose we have simulated $\{(W_{i,k}^m : 0 \leq k < 2^m) : m \leq N\}$ for some N (to be discussed momentarily) and define

$$\tau_1(N) = \inf\{n \geq N + 1 : |L_{i,j}^n(m) - L_{i,j}^n(l)| > (m - l)^\beta \Delta_n^{2\alpha} \text{ for some } 0 \leq l < m < 2^{n-1}\}.$$

Because of Lemma 4 we have that the event $\{\tau_1(N) = \infty\}$ has positive probability. We will explain how to simulate a Bernoulli random variable with success parameter $P(\tau_1(N) = \infty | \mathcal{F}_N)$. If such Bernoulli represents a success, then we have that $N_2 = N$ and we would have basically concluded the difficult part of the simulation procedure (the rest would be simulating under a series of conditioning events whose probability increases to one n grows). If the Bernoulli in question represents a failure (i.e its value is zero), then we will try again until obtaining a successful Bernoulli trial.

Now, part of the problem is that Algorithm I has been already executed, so $N \geq N_1$, in other words, while the random variables $\{W_{i,k}^n : 0 \leq k < 2^n\}$ are independent (for fixed $n > N$), they are no longer identically distributed. Instead, $W_{i,k}^n$ is standard Gaussian conditioned on the event $\{|W_{i,k}^n| < 4\sqrt{n+1}\}$.

Nevertheless, if n is large enough, all of the events $\{|W_{i,k}^n| < 4\sqrt{n+1}\}$ will occur with high probability. So, we shall first proceed to explain how to simulate a Bernoulli random variable with probability of success $P(\tau_1(n') = \infty | \mathcal{F}_{n'})$ assuming n' is a deterministic number. The procedure

actually will produce both the outcome of the Bernoulli trial and if such outcome is a failure (i.e. $\tau_1(n') < \infty$), also

$$\{W_{i,k}^m : 1 \leq k \leq 2^m, n' < m \leq \tau_1(n')\}.$$

Our procedure is based on acceptance / rejection using a carefully chosen proposal distribution for the $W_{i,k}^m$'s based on exponential tilting of the $L_{i,j}^{n'}(k)$'s, conditional on $\mathcal{F}_{n'}$. To this end, we will need to compute the associated (conditional on $\mathcal{F}_{n'}$) moment generating function of $L_{i,j}^n(k)$ and the family of distributions induced over the $W_{i,k}^n$'s and $W_{j,k}^n$'s when exponentially tilting $L_{i,j}^{n'}(k)$, this will be done in Section 3.5.1. Then, we need some large deviations estimates in order to enforce the feasibility of a certain randomization procedure, these estimates are given in Section 3.5.3. These are all the elements needed for the simulation procedure of α -Hölder Norms of Lévy Areas given in Section 3.6.

3.5.1. Basic Notation, Conditional Moment Generating Functions, and Associated Exponential Tilt-ing. First, we recall the wavelet synthesis discussed in Section 3.1, which was explained for a single Brownian motion. Since we will work with d' Brownian motions here we need to adapt the notation. For each $i \in \{1, \dots, d'\}$ let $\{(W_{i,k}^n : 1 \leq k \leq 2^n) : n \geq 1\}$ be the sequence of i.i.d. $N(0,1)$ random variables arising in the wavelet synthesis (9) for $Z_i(\cdot)$.

Now, define

$$\mathcal{F}_n = \sigma\{(W_{i,k}^m : 0 \leq k < 2^m) : m \leq n\}.$$

and for the conditional expectation given \mathcal{F}_n we write

$$E_n(\cdot) := E(\cdot | \mathcal{F}_n).$$

In order to reduce the length of some of the equations that follow, we write, for each $r \in \{1, 2, \dots, 2^n\}$,

$$(17) \quad \Lambda_i^n(t_r^n) := (Z_i(t_r^n) - Z_i(t_{r-1}^n)).$$

Then, using the following very useful pair of equations (for $k = 1, 2, \dots, 2^{n-1}$)

$$(18) \quad \begin{aligned} \Lambda_i^n(t_{2k-1}^n) &= \frac{1}{2} \Lambda_i^{n-1}(t_k^{n-1}) + \Delta_n^{1/2} W_{i,k}^n. \\ \Lambda_i^n(t_{2k}^n) &= \frac{1}{2} \Lambda_i^{n-1}(t_k^{n-1}) - \Delta_n^{1/2} W_{i,k}^n, \end{aligned}$$

we can see that

$$\mathcal{F}_n = \sigma\{\cup_{m \leq n} (Z(t) - Z(s)) : 0 \leq s < t \leq 1, t, s \in D_m\}.$$

and we have that (for $0 \leq k \leq 2^{n-1}$)

$$L_{i,j}^n(k) = \sum_{r=1}^k \Lambda_i^n(t_{2r-1}^n) \Lambda_j^n(t_{2r}^n).$$

Assume that $k < k'$, we will iteratively compute

$$(19) \quad \begin{aligned} E_n\{\exp(\theta_0\{L_{i,j}^{n+m}(k') - L_{i,j}^{n+m}(k)\})\} \\ = E_n[E_{n+1}[\dots E_{n+m-1}[\exp(\theta_0\{L_{i,j}^{n+m}(k') - L_{i,j}^{n+m}(k)\})]\dots]]. \end{aligned}$$

We first start from inner expectation.

Corollary 2.

$$\begin{aligned} & E_{n+m-1} \exp(\theta_0 \Lambda_i^{n+m} (t_{2r-1}^{n+m}) \Lambda_j (t_{2r}^{n+m})) \\ &= (1 - \theta_0^2 \Delta_{n+m}^2)^{-1/2} \exp \left(\theta_1 \Lambda_j^{n+m-1} (t_r^{n+m-1}) \Lambda_i^{n+m-1} (t_r^{n+m-1}) \right) \\ & \exp \left(\eta_1 \Lambda_j^{n+m-1} (t_r^{n+m-1})^2 + \eta_1 \Lambda_i^{n+m-1} (t_r^{n+m-1})^2 \right), \end{aligned}$$

where

$$\theta_1 := \theta_0 (1 - \theta_0^2 \Delta_{n+m}^2)^{-1} / 4, \quad \eta_1 := \theta_0^2 (1 - \theta_0^2 \Delta_{n+m}^2)^{-1} \Delta_{n+m} / 8.$$

Moreover, define

$$\begin{aligned} & P'_{n+m, t_r^{n+m}} \left(W_{i,r}^{n+m} \in A, W_{j,r}^{n+m} \in B \right) \\ &= \frac{E_{n+m-1} \left(I(W_{i,r}^{n+m} \in A, W_{j,r}^{n+m} \in B) \exp(\theta_0 \Lambda_i^{n+m} (t_{2r-1}^{n+m}) \Lambda_j^{n+m} (t_{2r}^{n+m})) \right)}{E_{n+m-1} \exp(\theta_0 \Lambda_i^{n+m} (t_{2r-1}^{n+m}) \Lambda_j^{n+m} (t_{2r}^{n+m}))}, \end{aligned}$$

then under $P'_{n+m, t_r^{n+m}}$, and given \mathcal{F}_{n+m-1} , we have that $(W_{i,r}^{n+m}, W_{j,r}^{n+m})$ follows a Gaussian distribution with covariance matrix

$$\Sigma_{n+m}^{i,j} (t_r^{n+m}) = \frac{1}{1 - \theta_0^2 \Delta_{n+m}^2} \begin{pmatrix} 1 & -\theta_0 \Delta_{n+m} \\ -\theta_0 \Delta_{n+m} & 1 \end{pmatrix},$$

and mean vector

$$\mu_{n+m}^{i,j} (t_r^{n+m}) = \Sigma_{n+m}^{i,j} (t_r^{n+m}) \begin{pmatrix} \theta_0 \Delta_{n+m}^{1/2} \Lambda_j^{n+m-1} (t_r^{n+m-1}) / 2 \\ -\theta_0 \Delta_{n+m}^{1/2} \Lambda_j^{n+m-1} (t_r^{n+m-1}) / 2 \end{pmatrix}.$$

So, from Corollary 2 we conclude that

$$\begin{aligned} & E_{n+m-1} [\exp(\theta_0 \sum_{r=k+1}^{k'} \Lambda_i^{n+m} (t_{2r-1}^{n+m}) \Lambda_j (t_{2r}^{n+m}))] \\ &= (1 - \theta_0^2 \Delta_{n+m}^2)^{-(k'-k)/2} \exp(\theta_1 \sum_{r=k+1}^{k'} \Lambda_j^{n+m-1} (t_r^{n+m-1}) \Lambda_i^{n+m-1} (t_r^{n+m-1})) \\ & \exp(\eta_1 \sum_{r=k+1}^{k'} \Lambda_j^{n+m-1} (t_r^{n+m-1})^2 + \eta_1 \sum_{r=k+1}^{k'} \Lambda_i^{n+m-1} (t_r^{n+m-1})^2). \end{aligned} \tag{20}$$

If $m \geq 2$, we can continue taking the corresponding conditional expectation given \mathcal{F}_{n+m-2} . Due to the recursive nature of (19) and the linear quadratic terms that arise in (20) it is convenient to consider

$$\begin{aligned} & \sum_{r=1}^{2^{n+m-1}} \theta_1 (t_r^{n+m-1}) \Lambda_j^{n+m-1} (t_r^{n+m-1}) \Lambda_i^{n+m-1} (t_r^{n+m-1}) \\ & + \sum_{r=1}^{2^{n+m-1}} \eta_1 (t_r^{n+m-1}) (\Lambda_j^{n+m-1} (t_r^{n+m-1})^2 + \Lambda_i^{n+m-1} (t_r^{n+m-1})^2), \end{aligned} \tag{21}$$

where

$$\theta_1 (t_r^{n+m-1}) = \theta_1 \times I(r \in \{k+1, \dots, k'\}), \quad \eta_1 (t_r^{n+m-1}) = \eta_1 \times I(r \in \{k+1, \dots, k'\}).$$

Then, recursively define for $l = 2, \dots, m$

(22)

$$\begin{aligned}\theta_+^l(t_r^{m+n-l}) &= \theta_{l-1}(t_{2r-1}^{m+n-l+1}) + \theta_{l-1}(t_{2r}^{m+n-l+1}), \theta_-^l(t_r^{m+n-l}) = \theta_{l-1}(t_{2r-1}^{m+n-l+1}) - \theta_{l-1}(t_{2r}^{m+n-l+1}) \\ \eta_+^l(t_r^{m+n-l}) &= \eta_{l-1}(t_{2r-1}^{m+n-l+1}) + \eta_{l-1}(t_{2r}^{m+n-l+1}), \eta_-^l(t_r^{m+n-l}) = \eta_{l-1}(t_{2r-1}^{m+n-l+1}) - \eta_{l-1}(t_{2r}^{m+n-l+1}) \\ \rho_l(t_r^{m+n-l}) &= \frac{\Delta_{n+m-l+1}\theta_+^l(t_r^{m+n-l})}{1 - 2\Delta_{n+m-l+1}\eta_+^l(t_r^{m+n-l})}, \\ h_l(t_r^{m+n-l}) &= \frac{\Delta_{n+m-l+1}}{(1 - 2\Delta_{n+m-l+1}\eta_+^l(t_r^{m+n-l})) (1 - \rho_l(t_r^{m+n-l})^2)},\end{aligned}$$

and set

$$\begin{aligned}\eta_l(t_r^{m+n-l}) &= \frac{\eta_+^l(t_r^{m+n-l})}{4} \\ &+ \frac{h_l(t_r^{m+n-l})}{8} \left(\theta_-^l(t_r^{m+n-l})^2 + 4\eta_-^l(t_r^{m+n-l})^2 + 4\theta_-^l(t_r^{m+n-l})\eta_-^l(t_r^{m+n-l})\rho_l(t_r^{m+n-l}) \right), \\ \theta_l(t_r^{m+n-l}) &= \frac{\theta_+^l(t_r^{m+n-l})}{4} \\ &+ h_l(t_r^{m+n-l}) \left(\theta_-^l(t_r^{m+n-l})\eta_-^l(t_r^{m+n-l}) + \frac{1}{4}\theta_-^l(t_r^{m+n-l})^2 g_l(t_r^{m+n-l}) + \eta_-^l(t_r^{m+n-l})^2 \rho_l(t_r^{m+n-l}) \right).\end{aligned}$$

Finally, define

$$\begin{aligned}A(t_r^{n+m-l}) &= \theta_{l-1}(t_{2r-1}^{n+m-l+1}) \Lambda_j^{n+m-l+1}(t_{2r-1}^{n+m-l+1}) \Lambda_i^{n+m-l+1}(t_{2r-1}^{n+m-l+1}) \\ &+ \theta_{l-1}(t_{2r}^{n+m-l+1}) \Lambda_j^{n+m-l+1}(t_{2r}^{n+m-l+1}) \Lambda_i^{n+m-l+1}(t_{2r}^{n+m-l+1}), \\ B(t_r^{n+m-l}) &= \eta_{l-1}(t_{2r-1}^{n+m-l+1}) (\Lambda_j^{n+m-l+1}(t_{2r-1}^{n+m-l+1})^2 + \Lambda_j^{n+m-l+1}(t_{2r-1}^{n+m-l+1})^2) \\ &+ \eta_{l-1}(t_{2r}^{n+m-l+1}) (\Lambda_j^{n+m-l+1}(t_{2r}^{n+m-l+1})^2 + \Lambda_j^{n+m-l+1}(t_{2r}^{n+m-l+1})^2),\end{aligned}$$

and

$$C(t_r^{n+m-l}) = \left(1 - 2\Delta_{n+m-l}\eta_+^l(t_r^{m+n-l})\right)^{-1} \left(1 - \rho_l(t_r^{m+n-l})^2\right)^{-1/2}.$$

So, in particular we can write (21) as

$$\sum_{r=1}^{2^{n+m-2}} (A(t_r^{n+m-2}) + B(t_r^{n+m-2})),$$

and the following result is key in evaluating (19).

Corollary 3. For $l = 2, 3, \dots, m$ and $r = 1, 2, \dots, 2^{n+m-l}$

$$\begin{aligned} & E_{n+m-l} \exp(A(t_r^{n+m-l}) + B(t_r^{n+m-l})) \\ &= C(t_r^{n+m-l}) \exp\left(\theta_l(t_r^{m+n-l}) \Lambda_i(t_r^{m+n-l}) \Lambda_j(t_r^{m+n-l})\right) \\ & \quad \exp\left(\eta_l(t_r^{m+n-l}) \left(\Lambda_i(t_r^{m+n-l})^2 + \Lambda_j(t_r^{m+n-l})^2\right)\right). \end{aligned}$$

Moreover, define

$$\begin{aligned} & P'_{n+m-l+1, t_r^{n+m-l+1}} \left(W_{i,r}^{n+m-l+1} \in A, W_{j,r}^{n+m-l+1} \in B \right) \\ &= \frac{E_{n+m-l} \left(I(W_{i,r}^{n+m-l+1} \in A, W_{j,r}^{n+m-l+1} \in B) \exp(A(t_r^{n+m-l}) + B(t_r^{n+m-l})) \right)}{E_{n+m-l} \exp(A(t_r^{n+m-l}) + B(t_r^{n+m-l}))}, \end{aligned}$$

then under $P'_{n+m-l+1, t_r^{n+m-l+1}}$, and given \mathcal{F}_{n+m-l} , we have that $(W_{i,r}^{n+m-l+1}, W_{j,r}^{n+m-l+1})$ follows a Gaussian distribution with covariance matrix

$$\begin{aligned} & \Sigma_{n+m-l+1}^{i,j} \left(t_r^{n+m-l+1} \right) \\ &= \frac{1}{1 - \rho_l(t_r^{m+n-l})^2} \begin{pmatrix} (1 - 2\Delta_{n+m-l}\eta_+^l(t_r^{m+n-l}))^{-1} & g_l(t_r^{m+n-l}) \\ g_l(t_r^{m+n-l}) & (1 - 2\Delta_{n+m-l}\eta_+^l(t_r^{m+n-l}))^{-1} \end{pmatrix} \end{aligned}$$

and mean vector

$$\begin{aligned} & \mu_{n+m}^{i,j} \left(t_r^{n+m-l+1} \right) \\ &= \Delta_{n+m-l}^{1/2} \Sigma_{n+m-l+1}^{i,j} \left(t_r^{n+m-l+1} \right) \begin{pmatrix} \Lambda_i(t_r^{n+m-l}) \eta_-^l(t_r^{n+m-l}) + \frac{1}{2} \Lambda_j(t_r^{n+m-l}) \theta_-^l(t_r^{n+m-l}) \\ \Lambda_j(t_r^{n+m-l}) \eta_-^l(t_r^{n+m-l}) + \frac{1}{2} \Lambda_i(t_r^{n+m-l}) \theta_-^l(t_r^{n+m-l}) \end{pmatrix}. \end{aligned}$$

Using Corollary 3 we conclude that

$$\begin{aligned} & E_{n+m-l} \exp\left(\sum_{r=1}^{2^{n+m-l}} (A(t_r^{n+m-l}) + B(t_r^{n+m-l}))\right) \\ &= \prod_{r=1}^{2^{n+m-l}} C(t_r^{n+m-l}) \times \exp\left(\sum_{r=1}^{2^{n+m-l-1}} (A(t_r^{n+m-l-1}) + B(t_r^{n+m-l-1}))\right). \end{aligned}$$

Therefore, combining Corollary 2 and repeatedly iterating the previous expression we conclude that

$$\begin{aligned} & E_n \exp(\theta_0 \{L_{i,j}^{n+m}(k) - L_{i,j}^{n+m}(k')\}) \\ &= (1 - \theta_0^2 \Delta_{n+m}^2)^{-(k'-k)/2} \prod_{l=2}^m \prod_{r=1}^{2^{n+m-l}} C(t_r^{n+m-l}) \\ (23) \quad & \times \exp\left(\sum_{r=1}^{2^n} \theta_m(t_r^n) \Lambda_i(t_r^n) \Lambda_j(t_r^n) + \sum_{r=1}^{2^n} \eta_m(t_r^n) \{\Lambda_i(t_r^n)^2 + \Lambda_j(t_r^n)^2\}\right). \end{aligned}$$

3.5.2. *Appendix to Section 3.5.1: Technical Development.* We first provide the following auxiliary result which summarizes basic computations of moment generating functions of quadratic forms of bivariate Gaussian random variables.

Lemma 6. *Suppose that Y and Z are i.i.d. $N(0, 1)$ random variables, then for any numbers $a_1, a_2, b, c_1, c_2 \in \mathbb{R}$ define*

$$\phi(a, b, c) := E \exp(a_1 Y + a_2 Z + b Y Z + c_1 Y^2 + c_2 Z^2),$$

then we have that if $|2c_i| < 1$ for $i = 1, 2$, and $|b| < (1 - 2c_1)(1 - 2c_2)$

$$\begin{aligned} \phi(a, b, c) &= (1 - 2c_1)^{-1/2} (1 - 2c_2)^{-1/2} \left(1 - (b(1 - 2c_1)^{-1/2} (1 - 2c_2)^{-1/2})^2 \right)^{-1/2} \\ &\times \exp \left(\frac{a_1^2 (1 - 2c_1)^{-1} + a_2^2 (1 - 2c_2)^{-1} + 2a_1 a_2 b (1 - 2c_1)^{-1} (1 - 2c_2)^{-1}}{2(1 - b^2 (1 - 2c_1)^{-1} (1 - 2c_2)^{-1})} \right) \end{aligned}$$

Moreover, if we let

$$P'(Y \in dy, Z \in dz) = P(Y \in dy, Z \in dz) \frac{\exp(a_1 y + a_2 z + b y z + c_1 y^2 + c_2 z^2)}{\phi(\theta; a, b, c)},$$

then under $P'(\cdot)$ we have that (Y, Z) are distributed bivariate Gaussian with covariance matrix

$$\begin{aligned} \Sigma(a, b, c) &= \frac{1}{1 - b^2 (1 - 2c_1)^{-1} (1 - 2c_2)^{-1}} \begin{pmatrix} (1 - 2c_1)^{-1} & b(1 - 2c_1)^{-1/2} (1 - 2c_2)^{-1/2} \\ b(1 - 2c_1)^{-1/2} (1 - 2c_2)^{-1/2} & (1 - 2c_2)^{-1} \end{pmatrix}, \end{aligned}$$

and mean vector

$$\mu(a, b, c) = \Sigma(a, b, c) \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}.$$

Proof. First it follows easily that $E \exp(c_1 Y^2 + c_2 Z^2) = (1 - 2c_1)^{-1/2} (1 - 2c_2)^{-1/2}$, and under the probability measure

$$P_1(Y \in dy, Z \in dz) = \frac{\exp(c_1 y^2 + c_2 z^2)}{E \exp(c_1 Y^2 + c_2 Z^2)} P(Y \in dy) P(Z \in dz)$$

Y and Z are independent with distributions $N(0, (1 - 2c_1)^{-1})$ and $N(0, (1 - 2c_2)^{-1})$, respectively. Therefore,

$$\begin{aligned} \phi(a, b, c) &= (1 - 2c_1)^{-1/2} (1 - 2c_2)^{-1/2} E_1 \exp(a_1 Y + a_2 Z + b Y Z) \\ &= (1 - 2c_1)^{-1/2} (1 - 2c_2)^{-1/2} \\ &\times E \exp \left(a_1 Y (1 - 2c_1)^{-1/2} + a_2 Z (1 - 2c_2)^{-1/2} + b (1 - 2c_1)^{-1/2} (1 - 2c_2)^{-1/2} Y Z \right). \end{aligned}$$

Now, given $|\theta| < 1$ define $P_2(\cdot)$ via

$$P_2(Y \in dy, Z \in dz) = \frac{P(Y \in dy, Z \in dz) \exp(\chi y z)}{E \exp(\chi Y Z)}.$$

Observe that

$$P(Y \in dy, Z \in dz) \exp(\chi y z) = \frac{1}{2\pi} \exp(-y^2/2 - z^2/2 + \chi y z)$$

and

$$-y^2/2 - z^2/2 + \chi y z = -(y, z) \Sigma^{-1} \begin{pmatrix} y \\ z \end{pmatrix} / 2,$$

where

$$\Sigma^{-1} = \begin{pmatrix} 1 & -\chi \\ -\chi & 1 \end{pmatrix},$$

and thus

$$\Sigma = \frac{1}{1 - \chi^2} \begin{pmatrix} 1 & \chi \\ \chi & 1 \end{pmatrix}.$$

Therefore, under $P_2(\cdot)$, (Y, Z) is distributed bivariate normal with mean zero and covariance matrix Σ , with

$$\chi = b(1 - 2c_1)^{-1/2}(1 - 2c_2)^{-1/2}$$

and we also must have that if $|\chi| < 1$,

$$E \exp(\phi Y Z) = (1 - \chi^2)^{-1/2} = \left(1 - (b(1 - 2c_1)^{-1/2}(1 - 2c_2)^{-1/2})^2\right)^{-1/2}.$$

Consequently, we conclude that

$$\begin{aligned} \phi(a, b, c) &= (1 - 2c_1)^{-1/2}(1 - 2c_2)^{-1/2} \left(1 - (b(1 - 2c_1)^{-1/2}(1 - 2c_2)^{-1/2})^2\right)^{-1/2} \\ &\quad \times E_2 \exp(a_1 Y(1 - 2c_1)^{-1/2} + a_2 Z(1 - 2c_2)^{-1/2}). \end{aligned}$$

The final expression for $\phi(a, b, c)$ is obtained from the fact that

$$\begin{aligned} &E_2 \exp(a_1 Y(1 - 2c_1)^{-1/2} + a_2 Z(1 - 2c_2)^{-1/2}) \\ &= \exp\left(\text{Var}_2(a_1 Y(1 - 2c_1)^{-1/2} + a_2 Z(1 - 2c_2)^{-1/2})/2\right). \end{aligned}$$

And $P'(\cdot)$ is equivalent to a standard exponentially tilting to the measure $P_2(\cdot)$ using as the natural parameter the vector

$$(a_1(1 - 2c_1)^{-1/2}, a_2(1 - 2c_2)^{-1/2}),$$

and thus under $P'(\cdot)$ the covariance matrix is the same as under $P_2(\cdot)$ and the mean vector is equal to $\mu(a, b, c)$. \square

We now are ready to provide the proof of Corollary 2.

Proof of Corollary 2. Let us examine a term of the form $\Lambda_i^{n+m}(t_{2r-1}^{n+m}) \Lambda_j(t_{2r}^{n+m})$,

$$\begin{aligned} &\Lambda_i^{n+m}(t_{2r-1}^{n+m}) \Lambda_j(t_{2r}^{n+m}) \\ &= (\Lambda_i^{n+m-1}(t_r^{n+m-1})/2 + \Delta_{n+m}^{1/2} W_{i,r}^{n+m})(\Lambda_j^{n+m-1}(t_r^{n+m-1})/2 - \Delta_{n+m}^{1/2} W_{j,r}^{n+m}) \\ &= \Lambda_i^{n+m-1}(t_r^{n+m-1}) \Lambda_j^{n+m-1}(t_r^{n+m-1})/4 - \Delta_{n+m} W_{i,r}^{n+m} W_{j,r}^{n+m} \\ &\quad + \Delta_{n+m}^{1/2} W_{i,r}^{n+m} \Lambda_j^{n+m-1}(t_r^{n+m-1})/2 - \Delta_{n+m}^{1/2} W_{j,r}^{n+m} \Lambda_i^{n+m-1}(t_r^{n+m-1})/2. \end{aligned}$$

Then, we have that Corollary 2 follows immediately from Lemma 6. \square

Finally, we provide the proof of Corollary 3.

Proof of Corollary 3. Recall that for each $r \in \{1, 2, \dots, 2^n\}$,

$$\Lambda_i^n(t_r^n) := (Z_i(t_r^n) - Z_i(t_{r-1}^n)).$$

So

$$\begin{aligned} \Lambda_i^n(t_{2r-1}^n) &= \Lambda_i^n(t_r^{n-1})/2 + \Delta_n^{1/2} W_{i,r}^n, \\ \Lambda_i^n(t_{2r}^n) &= \Lambda_i^n(t_r^{n-1})/2 - \Delta_n^{1/2} W_{i,r}^n. \end{aligned}$$

We perform the first iteration in full detail, the rest are immediate just adjusting the notation. From Corollary 2 we obtain that

$$\begin{aligned} & E_{n+m-1} \exp \left(\theta_0 [L_{i,j}^{n+m}(k') - L_{i,j}^{n+m}(k)] \right) \\ &= \exp \left(\frac{1}{2} \sum_{r=k+1}^{k'} \frac{\theta_0^2 \Delta_{n+m}}{4(1 - \theta_0^2 \Delta_{n+m}^2)} \Lambda_i(t_r^{n+m-1})^2 + \frac{1}{2} \sum_{r=k+1}^{k'} \frac{\theta_0^2 \Delta_{n+m}}{4(1 - \theta_0^2 \Delta_{n+m}^2)} \Lambda_j(t_r^{n+m-1})^2 \right) \\ &\times \exp \left(\sum_{r=k+1}^{k'} \frac{\theta_0 \Delta_{n+m}}{4(1 - \theta_0^2 \Delta_{n+m}^2)} \Lambda_i(t_r^{n+m-1}) \Lambda_j(t_r^{n+m-1}) \right) \times (1 - \theta_0^2 \Delta_{n+m}^2)^{-(k'-k)/2}. \end{aligned}$$

Using the definitions in (22) we have that

$$\begin{aligned} & \frac{1}{2} \sum_{r=k+1}^{k'} \frac{\theta_0^2 \Delta_{n+m}}{4(1 - \theta_0^2 \Delta_{n+m}^2)} \Lambda_i(t_r^{n+m-1})^2 + \frac{1}{2} \sum_{r=k+1}^{k'} \frac{\theta_0^2 \Delta_{n+m}}{4(1 - \theta_0^2 \Delta_{n+m}^2)} \Lambda_j(t_r^{n+m-1})^2 \\ &+ \sum_{r=k+1}^{k'} \frac{\theta_0 \Delta_{n+m}}{4(1 - \theta_0^2 \Delta_{n+m}^2)} \Lambda_i(t_r^{n+m-1}) \Lambda_j(t_r^{n+m-1}) \end{aligned}$$

is equal to

$$\begin{aligned} & \sum_{r=1}^{2^{n+m-2}} (\eta_1(t_{2r-1}^{n+m-1}) \Lambda_i(t_{2r-1}^{n+m-1})^2 + \eta_1(t_{2r}^{n+m-1}) \Lambda_i(t_{2r}^{n+m-1})^2) \\ &+ \sum_{r=1}^{2^{n+m-2}} (\eta_1(t_{2r-1}^{n+m-1}) \Lambda_j(t_{2r-1}^{n+m-1})^2 + \eta_1(t_{2r}^{n+m-1}) \Lambda_j(t_{2r}^{n+m-1})^2) \\ &+ \sum_{r=1}^{2^{n+m-2}} (\theta_1(t_{2r-1}^{n+m-1}) \Lambda_i(t_{2r-1}^{n+m-1}) \Lambda_j(t_{2r-1}^{n+m-1}) + \theta_1(t_{2r}^{n+m-1}) \Lambda_i(t_{2r}^{n+m-1}) \Lambda_j(t_{2r}^{n+m-1})). \end{aligned}$$

We now expand each of the terms; to simplify the notation write

$$x = W_{i,r}^{n+m-1} \quad \text{and} \quad y = W_{j,r}^{n+m-1}.$$

Define $\sqrt{\Delta} = \Delta_{n+m-1}^{1/2}$, put $u = \Lambda_i(t_r^{n+m-2})$ and $v = \Lambda_j(t_r^{n+m-2})$

$$\begin{aligned} \Lambda_i(t_{2r-1}^{n+m-1}) &= u/2 + \sqrt{\Delta}x, & \Lambda_i(t_{2r}^{n+m-1}) &= u/2 - \sqrt{\Delta}x, \\ \Lambda_j(t_{2r-1}^{n+m-1}) &= v/2 + \sqrt{\Delta}y, & \Lambda_j(t_{2r}^{n+m-1}) &= v/2 - \sqrt{\Delta}y. \end{aligned}$$

Now, for brevity let us write $\eta_o = \eta_1(t_{2r-1}^{n+m-1})$ and $\eta_e = \eta_1(t_{2r}^{n+m-1})$ ('o' is used for odd, and 'e' for even)

$$\begin{aligned} & (\eta_1(t_{2r-1}^{n+m-1}) \Lambda_i(t_{2r-1}^{n+m-1})^2 + \eta_1(t_{2r}^{n+m-1}) \Lambda_i(t_{2r}^{n+m-1})^2) \\ &+ \eta_1(t_{2r-1}^{n+m-1}) \Lambda_j(t_{2r-1}^{n+m-1})^2 + \eta_1(t_{2r}^{n+m-1}) \Lambda_j(t_{2r}^{n+m-1})^2) \\ &= \left(\eta_o \left(u/2 + \sqrt{\Delta}x \right)^2 + \eta_e \left(u/2 - \sqrt{\Delta}x \right)^2 + \eta_o \left(v/2 + \sqrt{\Delta}y \right)^2 + \eta_e \left(v/2 - \sqrt{\Delta}y \right)^2 \right) \\ &= \frac{1}{4} u^2 (\eta_e + \eta_o) + \frac{1}{4} v^2 (\eta_e + \eta_o) + u(\eta_o - \eta_e) \sqrt{\Delta}x + v(\eta_o - \eta_e) \sqrt{\Delta}y + (\eta_e + \Delta \eta_o) \Delta x^2 + (\eta_e + \eta_o) \Delta y^2. \end{aligned}$$

Likewise, put $\theta_o = \theta_1(t_{2r-1}^{n+m-1})$ and $\theta_e = \theta_1(t_{2r}^{n+m-1})$

$$\begin{aligned} & (\theta_1(t_{2r-1}^{n+m-1}) \Lambda_i(t_{2r-1}^{n+m-1}) \Lambda_j(t_{2r-1}^{n+m-1}) + \theta_1(t_{2r}^{n+m-1}) \Lambda_i(t_{2r}^{n+m-1}) \Lambda_j(t_{2r}^{n+m-1})) \\ &= \theta_o \left(u/2 + \sqrt{\Delta}x \right) \left(v/2 + \sqrt{\Delta}y \right) + \theta_e \left(u/2 - \sqrt{\Delta}x \right) \left(v/2 - \sqrt{\Delta}y \right) \\ &= \frac{1}{4}uv(\theta_e + \theta_o) + (\theta_e + \theta_o)\Delta xy + \frac{1}{2}v(\theta_o - \theta_e)\sqrt{\Delta}x + \frac{1}{2}u(\theta_o - \theta_e)\sqrt{\Delta}y \end{aligned}$$

We then collect the terms free of x and y and obtain

$$\frac{u^2}{4}(\eta_e + \eta_o) + \frac{v^2}{4}(\eta_e + \eta_o) + \frac{uv}{4}(\theta_e + \theta_o).$$

Now the coefficients of x, y, x^2, y^2 , and xy

$$\begin{aligned} & \{u(\eta_o - \eta_e) + \frac{1}{2}v(\theta_o - \theta_e)\}\sqrt{\Delta}x + \{v(\eta_o - \eta_e) + \frac{1}{2}u(\theta_o - \theta_e)\}\sqrt{\Delta}y \\ &+ (\eta_e + \eta_o)\Delta x^2 + (\eta_e + \eta_o)\Delta y^2 \\ &+ (\theta_e + \theta_o)\Delta xy. \end{aligned}$$

And finally we can apply Lemma 6 to get the corresponding results. \square

3.5.3. Conditional Large Deviations Estimates for $L_{i,j}^n(k)$. We wish to estimate, for $k' > k$ and $k', k \in \{0, 1, \dots, 2^{n+m-1}\}$,

$$\begin{aligned} & P_n \left(|L_{i,j}^{n+m}(k') - L_{i,j}^{n+m}(k)| > (k' - k)^\beta \Delta_{n+m}^{2\alpha} \right) \\ & \leq \exp(-\theta_0 (k' - k)^\beta \Delta_{n+m}^{2\alpha}) \times \{E_n[\exp(\theta_0 \{L_{i,j}^{n+m}(k') - L_{i,j}^{n+m}(k)\})] \\ & + E_n[\exp(-\theta_0 \{L_{i,j}^{n+m}(k') - L_{i,j}^{n+m}(k)\})]\}. \end{aligned}$$

We borrow some intuition from the proof of Lemma 4 and select

$$(24) \quad \theta_0(m, k', k) := \theta_0 = \frac{\gamma}{(k' - k)^{1/2} \Delta_n^{2\alpha'} \Delta_m}.$$

We will drop the dependence on (m, k', k) for brevity. In addition, we pick $\gamma \leq 1/4$ and $\alpha' \in (\alpha, 1/2)$ so that

$$\exp(-\theta_0 (k' - k)^\beta \Delta_{n+m}^{2\alpha}) = \exp(-\gamma (k' - k)^{\beta-1/2} \Delta_n^{2(\alpha-\alpha')} \Delta_m^{2\alpha-1})$$

Our next task is to control the $E_n \exp(\theta_0 \{L_{i,j}^{n+m}(k') - L_{i,j}^{n+m}(k)\})$, which is the purpose of the following result, proved in the appendix to this section.

Lemma 7. *Suppose that θ_0 is chosen according to (24), and n is such that for $\varepsilon_0 \in (0, 1/2)$*

$$(25) \quad \max_{r \leq 2^n} \{|\Lambda_i(t_r^n)|, |\Lambda_j(t_r^n)|\} \leq \Delta_n^{\alpha'}$$

and

$$(26) \quad \left| \sum_{r=l+1}^m \Lambda_i(t_r^n) \Lambda_j(t_r^n) \right| \leq \varepsilon_0(m-l)^\beta \Delta_n^{2\alpha'} \text{ for all } 0 \leq l < m \leq 2^n$$

with $\alpha' \in (\alpha, 1/2)$, then

$$E_n[\exp(\theta_0 \{L_{i,j}^{n+m}(k') - L_{i,j}^{n+m}(k)\})] \leq 4 \exp\left(\varepsilon_0 \gamma (k' - k)^{\beta-1/2}\right).$$

Remark: It is very important to note that due to Lemma 2 we can always continue simulating the $W_{i,k}^m$'s (maybe conditional on $\{|W_{i,k}^m| < 4\sqrt{m+1}\}$ in case $m > N_1$) to make sure that (25) holds for some n . Similarly, condition (26) can be simultaneously enforced with (25) because of Lemma 4. Actually, Lemmas 2 and Lemma 4 indicate that conditions (25) and (26) will occur eventually for all n larger than some random threshold enough. Our simulation algorithms will ultimately detect such threshold, but Lemma 7 does not require that we know that threshold.

As a consequence of Lemma 7, using Chernoff's bound, we obtain the following proposition.

Proposition 1. *If n is such that (25) and (26) hold, then*

$$P_n \left(|L_{i,j}^{n+m}(k') - L_{i,j}^{n+m}(k)| > (k' - k)^\beta \Delta_{n+m}^{2\alpha} \right) \leq 8 \exp \left(-\frac{1}{2} \gamma (k' - k)^{\beta-1/2} \Delta_n^{2(\alpha-\alpha')} \Delta_m^{2\alpha-1} \right).$$

3.5.4. *Appendix to Section 3.5.3: Technical Development.* We now provide the proof of Lemma 7.

Proof of Lemma 7. Recalling expression (23), we establish the bound for $E_n \exp \left(\theta_0 \{L_{i,j}^{n+1}(k') - L_{i,j}^{n+m}(k)\} \right)$ by controlling the contribution of the term

$$(27) \quad \prod_{l=2}^m \prod_{r=1}^{2^{n+m-l}} C \left(t_r^{n+m-l} \right).$$

and the exponential term

$$(28) \quad \exp \left(\sum_{r=1}^{2^n} \theta_m(t_r^n) \Lambda_i(t_r^n) \Lambda_j(t_r^n) + \sum_{r=1}^{2^n} \eta_m(t_r^n) (\Lambda_i(t_r^n)^2 + \Lambda_j(t_r^n)^2) \right)$$

separately.

We start by analyzing θ_l and η_l . From Corollary 2, we have

$$\theta_1 = \frac{\theta_0}{4(1 - \theta_0^2 \Delta_{n+m}^2)} \text{ and } \eta_1 = \frac{\theta_0^2 \Delta_{n+m}}{8(1 - \theta_0^2 \Delta_{n+m}^2)}.$$

We notice that $2\eta_1 \leq \theta_1^2 \Delta_{n+m} \leq (5/2)\eta_1$.

Let

$$u = \max\{h : k' - k > 2^h\}.$$

We also denote

$$\underline{b} := \min\{r : \theta_l(t_r^{n+m-l}) > 0\}$$

and

$$\bar{b} := \max\{r : \theta_l(t_r^{n+m-l}) > 0\}.$$

The strategy throughout the rest of the proof proceeds as follows. We have that the $\theta_l(t_r^{n+m-l})$'s and $\eta_l(t_r^{n+m-l})$'s, $r = 1, 2, \dots, 2^{n+m-l}$, are nonnegative. We also have that for $l \leq u \wedge m$, the number of positive $\theta_l(t_r^{n+m-l})$'s and $\eta_l(t_r^{n+m-l})$'s reduces by about a half at each step l and also the actual value of the positive $\theta_l(t_r^{n+m-l})$'s and $\eta_l(t_r^{n+m-l})$'s shrinks by at least $1/2$. We will establish that if $m > u$, for $u < l \leq m$, there are at most two positive $\theta_l(t_r^{n+m-l})$'s and two positive $\eta_l(t_r^{n+m-l})$'s and at each step l , their values shrink by more than $2^{-3/2}$. Using these observations we will establish some facts and then use them to estimate (27) and finally (28). We now proceed to carry out this strategy.

We first verify the following claims.

Claim 1:

For $l \leq u$, we claim that $\theta_l(t_r^{n+m-l}), \eta_l(t_r^{n+m-l}) \geq 0$ for all $r = 1, 2, \dots, 2^{n+m-l}$ and $\theta_l(t_r^{n+m-l})$'s are equal for $r \in (\underline{b}, \bar{b})$ and we denote their values as θ_l . So, following the recursion in (22) we have that $\theta_l = \Delta_{l-1}\theta_1$. If $\theta_l(t_{\underline{b}}^{n+m-l}) \neq \theta_l(t_{\underline{b}+1}^{n+m-l})$, then $\theta_l(t_{\underline{b}}^{n+m-l}) < \theta_l(t_{\underline{b}+1}^{n+m-l}) = \theta_l$, and if $\theta_l(t_{\bar{b}}^{n+m-l}) \neq \theta_l(t_{\bar{b}-1}^{n+m-l})$, then $\theta_l(t_{\bar{b}}^{n+m-l}) < \theta_l(t_{\bar{b}-1}^{n+m-l}) = \theta_l$.

Likewise, $\eta_l(t_r^{n+m-l})$'s are equal for $r \in (\underline{b}, \bar{b})$; we denote their common values as η_l and we have from (22) that $\eta_l = \Delta_{l-1}\eta_1$. If $\eta_l(t_{\underline{b}}^{n+m-l}) \neq \eta_l(t_{\underline{b}+1}^{n+m-l})$, then $\eta_l(t_{\underline{b}}^{n+m-l}) < \eta_l(t_{\underline{b}+1}^{n+m-l})$, and if $\eta_l(t_{\bar{b}}^{n+m-l}) \neq \eta_l(t_{\bar{b}-1}^{n+m-l})$, then $\eta_l(t_{\bar{b}}^{n+m-l}) < \eta_l(t_{\bar{b}-1}^{n+m-l})$. In other words, at each step, l for $l < u$, $\theta_l(t_r^{n+m-l})$ and $\eta_l(t_r^{n+m-l})$ decay at rate $1/2$ if it is not at the boundary ($r \in (\underline{b}, \bar{b})$), and the boundary ones ($\theta_l(t_{\underline{b}}^{n+m-l}), \theta_l(t_{\bar{b}}^{n+m-l})$ and $\eta_l(t_{\underline{b}}^{n+m-l}), \eta_l(t_{\bar{b}}^{n+m-l})$), may decay at a faster rate.

We now prove the claim by induction using the recursive relation in (22). The claim is immediate for θ_1 and η_1 . Now suppose it holds for $\theta_l(t_r^{n+m-l})$ and $\eta_l(t_r^{n+m-l})$, $r = 1, 2, \dots, 2^{n+m-l}$. We next show that the claim holds for $\theta_{l+1}(t_r^{n+m-l-1})$, $r = 1, 2, \dots, 2^{n+m-l-1}$, as well. We omit the proof of $\eta_{l+1}(t_r^{n+m-l-1})$ here, as it follows exactly the same line of analysis as $\theta_{l+1}(t_r^{n+m-l-1})$.

We next divide the analysis into five cases.

Case 1. $\theta_l(t_{2r-1}^{m+n-l}) = \theta_l(t_{2r}^{m+n-l})$ and $\eta_l(t_{2r-1}^{m+n-l}) = \eta_l(t_{2r}^{m+n-l})$. Then $\theta_+^{l+1}(t_r^{m+n-l}) = 2\theta_l(t_{2r-1}^{m+n-l+1})$ and $\theta_-^{l+1}(t_r^{m+n-l}) = 0$. Likewise $\eta_+^{l+1}(t_r^{m+n-l}) = 2\eta_l(t_{2r-1}^{m+n-l+1})$ and $\eta_-^{l+1}(t_r^{m+n-l}) = 0$. From (22), we have $\theta_l(t_r^{m+n-l-1}) = \theta_{l-1}(t_{2r-1}^{m+n-l+1})/2$ and $\eta_l(t_r^{m+n-l-1}) = \eta_{l-1}(t_{2r-1}^{m+n-l+1})/2$.

Case 2. $\theta_l(t_{2r-1}^{m+n-l}) = 0$, $\theta_l(t_{2r}^{m+n-l}) > 0$ and $\eta_l(t_{2r-1}^{m+n-l}) = 0$, $\eta_l(t_{2r}^{m+n-l}) > 0$. Then we know that $2r = \underline{b}$. We also have $\theta_+^{l+1}(t_r^{m+n-l-1}) = \theta_l(t_{2r}^{m+n-l})$ and $\theta_-^{l+1}(t_r^{m+n-l-1}) = -\theta_l(t_{2r}^{m+n-l})$. Likewise, $\eta_+^{l+1}(t_r^{m+n-l-1}) = \eta_l(t_{2r}^{m+n-l})$ and $\eta_-^{l+1}(t_r^{m+n-l-1}) = -\eta_l(t_{2r}^{m+n-l})$. We rewrite the expression for $\theta_{l+1}(t_r^{n+m-l-1})$ in (22) as

$$\begin{aligned} \theta_{l+1}(t_r^{m+n-l-1}) &= \theta_+^{l+1}(t_r^{m+n-l-1})\frac{1}{4} + |\theta_-^{l+1}(t_r^{m+n-l-1})|\{h_{l+1}(t_r^{m+n-l-1})|\eta_-^{l+1}(t_r^{m+n-l-1})| \\ &\quad + \frac{1}{4}h_{l+1}(t_r^{m+n-l-1})|\theta_-^{l+1}(t_r^{m+n-l-1})|\rho_{l+1}(t_r^{m+n-l-1}) \\ &\quad + h_{l+1}(t_r^{m+n-l-1})\eta_-^{l+1}(t_r^{m+n-l-1})^2\frac{\rho_{l+1}(t_r^{m+n-l-1})}{|\theta_-^{l+1}(t_r^{m+n-l-1})|}\} \\ &= \theta_l(t_{2r}^{m+n-l})\{\frac{1}{4} + h_{l+1}(t_r^{m+n-l-1})\eta_l(t_{2r}^{m+n-l}) \\ &\quad + \frac{1}{4}h_{l+1}(t_r^{m+n-l-1})\theta_l(t_{2r}^{m+n-l})\rho_{l+1}(t_r^{m+n-l-1}) \\ &\quad + h_{l+1}(t_r^{m+n-l-1})\eta_l(t_{2r}^{m+n-l})^2\frac{\rho_{l+1}(t_r^{m+n-l-1})}{\theta_l(t_{2r}^{m+n-l})}\} \end{aligned}$$

As

$$\theta_l\Delta_{n+m-l} \leq \theta_1\Delta_{n+m-1} \leq \frac{1}{4}$$

and

$$\eta_l\Delta_{n+m-l} \leq \eta_1\Delta_{n+m-1} \leq \frac{1}{48},$$

it is then easy to check that

$$\frac{1}{4}\theta_l(t_{2r}^{m+n-l}) < \theta_{l+1}(t_r^{m+n-l-1}) < \frac{3}{10}\theta_l \leq \frac{3}{5}\theta_{l+1}.$$

Case 3. $\theta_l(t_{2r-1}^{m+n-l}) > 0$, $\theta_l(t_{2r}^{m+n-l}) = 0$ and $\eta_l(t_{2r-1}^{m+n-l}) > 0$, $\eta_l(t_{2r}^{m+n-l}) = 0$. Then we know that $2r - 1 = \bar{b}$. Following the same line of analysis as in Case 2, we have

$$\frac{1}{4}\theta_l(t_{2r}^{m+n-l}) < \theta_{l+1}(t_r^{m+n-l-1}) < \frac{3}{10}\theta_l \leq \frac{3}{5}\theta_{l+1}.$$

Case 4. $0 < \theta_l(t_{2r-1}^{m+n-l}) < \theta_l(t_{2r}^{m+n-l})$ and $0 < \eta_l(t_{2r-1}^{m+n-l}) < \theta_l(t_{2r}^{m+n-l})$. Then we know that $2r - 1 = \underline{b}$. There exist $\xi < 1$, such that $\theta_l(t_{2r-1}^{m+n-l}) \leq \xi\theta_l(t_{2r}^{m+n-l}) = \xi\Delta_{l-1}\theta_1$ and $\eta_l(t_{2r-1}^{m+n-l}) \leq \xi\eta_l(t_{2r}^{m+n-l}) = \xi\Delta_{l-1}\eta_1$. From (22), we have

$$\begin{aligned} \theta_{l+1}(t_r^{m+n-l-1}) &\leq \theta_+^{l+1}(t_r^{m+n-l-1}) \left\{ \frac{1}{4} + h_{l+1}(t_r^{m+n-l-1})\eta_-^{l+1}(t_r^{m+n-l-1})^2 \frac{\rho_{l+1}(t_r^{m+n-l-1})}{\theta_+^{l+1}(t_r^{m+n-l-1})} \right\} \\ &\quad + |\theta_-^{l+1}(t_r^{m+n-l-1})| \{ h_{l+1}(t_r^{m+n-l-1})|\eta_-^{l+1}(t_r^{m+n-l-1})| \\ &\quad + \frac{1}{4}h_{l+1}(t_r^{m+n-l-1})|\theta_-^{l+1}(t_r^{m+n-l-1})|\rho_{l+1}(t_r^{m+n-l-1}) \}. \end{aligned}$$

As $|\theta_-^{l+1}(t_r^{m+n-l-1})| \leq \theta_l$ and $|\eta_-^{l+1}(t_r^{m+n-l-1})| \leq \eta_l$, it is easy to check that

$$\theta_{l+1}(t_r^{m+n-l-1}) < \theta_+^{l+1}(t_r^{m+n-l-1}) \left(\frac{1}{4} + 0.01 \right) + |\theta_-^{l+1}(t_r^{m+n-l-1})| \times 0.05.$$

Since $\theta_{l+1}(t_r^{m+n-l-1}) + |\theta_-^{l+1}(t_r^{m+n-l-1})| = \theta_l$, we have

$$\theta_{l+1}(t_r^{m+n-l-1}) < \theta_l \left(\left(\frac{1}{4} + 0.01 - 0.05 \right) (1 + \xi) + 0.05 \right) = \frac{1}{2}\theta_l \left(\frac{1}{2} + 0.02 + 0.42\xi \right) < \frac{\theta_l}{2} = \theta_{l+1}.$$

Case 5. $\theta_l(t_{2r-1}^{m+n-l}) > \theta_l(t_{2r}^{m+n-l}) > 0$ and $0 < \eta_l(t_{2r-1}^{m+n-l}) > \theta_l(t_{2r}^{m+n-l}) > 0$. Then we know that $2r = \bar{b}$. Following the same line of analysis as in Case 4, we have

$$\theta_{l+1}(t_r^{m+n-l-1}) < \theta_{l+1}.$$

We thus prove that the claim holds for $\theta_{l+1}(t_r^{m+n-l-1})$, $r = 1, 2, \dots, 2^{n+m-l-1}$, as well.

We have established Claim 1. We now continue with a second claim.

Claim 2:

For $u < l < m$, we have at most two positive $\theta_l(t_r^{m+n-l})$'s, namely $\theta_l(t_{\underline{b}}^{m+n-l})$ and $\theta_l(t_{\bar{b}}^{m+n-l})$. Notice that it is possible that $\underline{b} = \bar{b}$. We then claim that if $\underline{b} \neq \bar{b}$, $\theta_l(t_{\underline{b}}^{m+n-l}) \leq \Delta_{l-1}\theta_1 2^{-(l-u-1)/2}$ and $\theta_l(t_{\bar{b}}^{m+n-l}) \leq \Delta_{l-1}\theta_1 2^{-(l-u-1)/2}$. Similarly $\eta_l(t_{\underline{b}}^{m+n-l}) \leq \Delta_{l-1}\eta_1 2^{-(l-u-1)/2}$ and $\eta_l(t_{\bar{b}}^{m+n-l}) \leq \Delta_{l-1}\eta_1 2^{-(l-u-1)/2}$. If $\underline{b} = \bar{b}$, $\theta_l(t_{\underline{b}}^{m+n-l}) \leq \Delta_{l-1}\theta_1 2^{-(l-u-2)/2}$, $\theta_l(t_{\bar{b}}^{m+n-l}) \leq \Delta_{l-1}\theta_1 2^{-(l-u-2)/2}$ and $\eta_l(t_{\underline{b}}^{m+n-l}) \leq \Delta_{l-1}\eta_1 2^{-(l-u-2)/2}$, $\eta_l(t_{\bar{b}}^{m+n-l}) \leq \Delta_{l-1}\eta_1 2^{-(l-u-2)/2}$.

We prove the claim by induction. We shall give the proof of $\theta_l(t_r^{m+n-l})$ only, as the proof of $\eta_l(t_r^{m+n-l})$ follows exactly the same line of analysis. For $l = u$, we have the following cases.

- i) $\bar{b} = \underline{b} + 2$, \underline{b} is odd. In this case, $\theta_{l+1}(t_{(\underline{b}+1)/2}^{m+n-l-1}) < \Delta_l \theta_1$, which follows from the analysis in Case 4 for $l \leq u$. And $\theta_{l+1}(t_{(\bar{b}+1)/2}^{m+n-l-1}) < (3/5)\Delta_l \theta_1$, following the analysis in Case 3 for $l \leq u$.
- ii) $\bar{b} = \underline{b} + 2$, \underline{b} is even. In this case, $\theta_{l+1}(t_{\underline{b}/2}^{m+n-l-1}) < (3/5)\Delta_l \theta_1$, which follows from the analysis in Case 2 for $l \leq u$. And $\theta_{l+1}(t_{\bar{b}/2}^{m+n-l-1}) < \Delta_l \theta_1$, following the analysis in Case 5, for $l \leq u$.
- iii) $\bar{b} = \underline{b} + 1$, \underline{b} is odd. In this case, let $\bar{\theta}_l = \max\{\theta_l(t_{\underline{b}}^{m+n-l}), \theta_l(t_{\bar{b}}^{m+n-l})\}$, Then following the same analysis as in Case 4 or Case 5 for $l \leq u$ (depending on which one of $\theta_l(t_{\underline{b}}^{m+n-l})$ and $\theta_l(t_{\bar{b}}^{m+n-l})$ is smaller), we have $\theta_{l+1}(t_{\bar{b}/2}^{m+n-l-1}) < \bar{\theta}_l/2 \leq \Delta_l \theta_1$.
- iv) $\bar{b} = \underline{b} + 1$, \underline{b} is even. In this case, $\theta_{l+1}(t_{\underline{b}/2}^{m+n-l-1}) < (3/5)\Delta_l \theta_1$, which follows from the analysis in Case 2 for $l \leq u$. And $\theta_{l+1}(t_{(\bar{b}+1)/2}^{m+n-l-1}) < (3/5)\Delta_l \theta_1$, following the analysis in Case 3 for $l \leq u$.

Therefore, the claim holds for $u + 1$. Suppose the claim holds for $l \geq u + 1$. Then when moving from level l to level $l + 1$, one of the following three cases can happen.

- a) $\bar{b} = \underline{b} + 1$ and \underline{b} is even. In this case, following the analysis in Case 2 and Case 3 for $l \leq u$, we have

$$\theta_{l+1}(t_{\underline{b}/2}^{m+n-l-1}) \leq \frac{3}{10}\theta_l(t_{\underline{b}}^{m+n-l}) \leq \Delta_l \theta_1 2^{-(l-u)/2}$$

and

$$\theta_{l+1}(t_{(\bar{b}+1)/2}^{m+n-l-1}) \leq \frac{3}{10}\theta_l(t_{\bar{b}}^{m+n-l}) \leq \Delta_l \theta_1 2^{-(l-u)/2}.$$

- b) $\bar{b} = \underline{b}$. In this case, following the analysis in Case 2 or Case 3 for $l \leq u$ (depending on whether \underline{b} is odd or even), we have

$$\theta_{l+1}(t_{[\underline{b}/2]}^{m+n-l-1}) \leq \frac{3}{10}\theta_l(t_{\underline{b}}^{m+n-l}) \leq \Delta_l \theta_1 2^{-(l-u-1)/2}.$$

- c) $\bar{b} = \underline{b} + 1$ and \underline{b} is odd. In this case, we let $\bar{\theta}_l = \max\{\theta_l(t_{\underline{b}}^{m+n-l}), \theta_l(t_{\bar{b}}^{m+n-l})\}$, Then we can use the same analysis as in Case 4 or Case 5 for $l \leq u$ (depending on which one of $\theta_l(t_{\underline{b}}^{m+n-l})$ and $\theta_l(t_{\bar{b}}^{m+n-l})$ is smaller) to conclude that

$$\theta_{l+1}(t_{\bar{b}/2}^{m+n-l-1}) < \frac{1}{2}\bar{\theta}_l \leq \Delta_l \theta_1 2^{-(l-u-1)/2}.$$

We notice that case c) can happen only once.

We are now ready to control the contribution of the term (27). As $\Delta_{n+m-l+1}\eta_+^l(t_r^{n+m-l}) \leq 1/30$ and $\rho_l(t_r^{n+m-l}) < 1/7$, we have when $m \leq u$

$$\begin{aligned} & \prod_{l=2}^m \prod_{r=1}^{2^{n+m-l}} C(t_r^{n+m-l}) \\ & \leq \prod_{l=2}^m \prod_{r=1}^{2^{n+m-l}} \exp \left(4\Delta_{n+m-l+1}\eta_+^l(t_r^{n+m-l}) + \rho_l(t_r^{n+m-l})^2 \right) \\ & \leq \prod_{l=2}^m \exp \left(\left(16\Delta_{n+m}\eta_1 + \frac{(4\Delta_{n+m}\theta_1)^2}{(1-8\Delta_{n+m}\eta_1)^2} \right) ((k' - k)\Delta_l + 2) \right) \\ & \leq \prod_{l=2}^m \exp \left(\left(\frac{11}{5} \frac{\gamma^2}{k' - k} \Delta_n^{1-2\alpha'} + \frac{6}{5} \frac{\gamma^2}{k' - k} \Delta_n^{2-4\alpha'} \right) ((k' - k)\Delta_l + 2) \right). \end{aligned}$$

The last inequality follows from Corollary 2 that $\theta_1 = \theta_0/4(1 - \theta_0^2\Delta_{n+m}^2)$, $\eta_1 = \theta_0^2\Delta_{n+m}/2(1 - \theta_0^2\Delta_{n+m}^2)$, and our choice of $\theta_0 = \gamma/((k'^{1/2}\Delta_n^{2\alpha'}\Delta_m))$. Then, as $(k' - k)^{-1} \leq 2^{-m}$,

$$\begin{aligned} & \prod_{l=2}^m \prod_{r=1}^{2^{n+m-l}} C(t_r^{n+m-l}) \\ & \leq \exp \left(\frac{11}{5} \gamma^2 \left(\sum_{l=2}^m \Delta_l + 2(m-1)\Delta_m \right) + \frac{6}{5} \gamma^2 \left(\sum_{l=2}^u \Delta_l + 2(m-1)\Delta_m \right) \right) \\ & \leq \exp \left(\frac{8}{25} \right). \end{aligned}$$

When $m > u$,

$$\begin{aligned} & \prod_{l=2}^m \prod_{r=1}^{2^{n+m-l}} C(t_r^{n+m-l}) \\ & \leq \prod_{l=2}^m \prod_{r=1}^{2^{n+m-l}} \exp \left(4\Delta_{n+m-l+1}\eta_+^l(t_r^{n+m-l}) + \rho_l(t_r^{n+m-l})^2 \right) \\ & \leq \prod_{l=2}^u \exp \left(\left(\frac{11}{5} \frac{\gamma^2}{k' - k} \Delta_n^{1-2\alpha'} + \frac{6}{5} \frac{\gamma^2}{k' - k} \Delta_n^{2-4\alpha'} \right) ((k' - k)\Delta_l + 2) \right) \\ & \quad \times \prod_{l=u+1}^m \exp \left(\frac{11}{5} \frac{\gamma^2}{k' - k} \Delta_n^{1-2\alpha'} \Delta_{l-u-2}^{1/2} + \frac{6}{5} \frac{\gamma^2}{k' - k} \Delta_n^{2-4\alpha'} \Delta_{l-u-2} \right). \end{aligned}$$

As $(k' - k)^{-1} \geq 2^{-u}$,

$$\begin{aligned} & \prod_{l=2}^m \prod_{r=1}^{2^{n+m-l}} C(t_r^{n+m-l}) \\ & \leq \exp \left(\frac{11}{5} \gamma^2 \left(\sum_{l=2}^u \Delta_l + 2(u-1)\Delta_u + \sum_{l=u+1}^m \Delta_{l-2}^{1/2} \right) + \frac{6}{5} \gamma^2 \left(\sum_{l=2}^u \Delta_l + 2(u-1)\Delta_u + \sum_{l=u+1}^m \Delta_{l-2} \right) \right) \\ & \leq \exp \left(\frac{1}{2} \right). \end{aligned}$$

For (28), we notice that under condition (25) and (26), we have

$$\begin{aligned} \left| \sum_{r=1}^{2^n} \theta_m(t_r^n) \Lambda_i(t_r^n) \Lambda_j(t_r^n) \right| &\leq \theta_1 \Delta_{m-1} \varepsilon_0 ((k' - k) \Delta_m)^\beta \Delta_n^{2\alpha'} + 2\theta_1 \Delta_{m-1} \Delta_n^{2\alpha'} \\ &\leq \varepsilon_0 \gamma (k' - k)^{\beta-1/2} + 2\gamma, \end{aligned}$$

and

$$\begin{aligned} \left| \sum_{r=1}^{2^n} \eta_m(t_r^n) (\Lambda_i(t_r^n)^2 + \Lambda_j(t_r^n)^2) \right| &\leq ((k' - k) \Delta_m + 2) \eta_1 \Delta_{m-1} 2\Delta_n^{2\alpha'} \\ &\leq 2\gamma^2. \end{aligned}$$

Combining the analysis for (27) and (28), we have

$$\begin{aligned} E_n \exp(\theta_0 \{L_{i,j}^{n+m}(k') - L_{i,j}^{n+m}(k)\}) &\leq \exp \left(\theta_0^2 \Delta_{n+m}^2 (k' - k) + \frac{1}{2} + \varepsilon_0 \gamma (k' - k)^{\beta-1/2} + 2\gamma + 2\gamma^2 \right) \\ &\leq 4 \exp \left(\varepsilon_0 \gamma (k' - k)^{\beta-1/2} \right). \end{aligned}$$

□

3.6. Joint Tolerance-Enforced Simulation for α -Hölder Norms and Proof of Theorem 2.

Define

$$\mathcal{C}_n(m) = \{|L_{i,j}^{n+m}(k') - L_{i,j}^{n+m}(k)| > (k' - k)^\beta \Delta_{n+m}^{2\alpha} \text{ for some } 0 \leq k < k' < 2^{n+m-1}\},$$

and put $\tau_1(n) = \inf\{m \geq 1 : \mathcal{C}_n(m) \text{ occurs}\}$. We write $\bar{\mathcal{C}}_n(m)$ for the complement of $\mathcal{C}_n(m)$, so that

$$P_n(\tau_1(n) < \infty) = \sum_{m=1}^{\infty} P(\mathcal{C}_n(m) \cap \cap_{l=1}^{m-1} \bar{\mathcal{C}}_n(m)).$$

To facilitate the explanation, we next introduce a few more notations. Let

$$\omega_{n:n+m} := \{W_{i,k}^l : 1 \leq k \leq 2^n, 1 \leq i \leq d', n < l \leq n+m\}.$$

In addition, define

$$v_n(k, k'|m) := 6 \exp \left(-\frac{1}{2} \gamma (k' - k)^{\beta-1/2} \Delta_n^{2(\alpha-\alpha')} \Delta_m^{2\alpha-1} \right) I(0 \leq k < k' \leq 2^{n+m-1}) I(m \geq 1)$$

$$b_n(m) := \sum_{0 \leq k < k' \leq 2^{n+m-1}} v_n(k, k'|m)$$

$$q_n(k, k'|m) := \frac{v_n(k, k'|m)}{b_n(m)}$$

and

$$P_{n,m}^{i,j,k,k'}(\omega_{n:n+m} \in \cdot) = \frac{E_n I(\omega_{n:n+m} \in \cdot) \exp \left(\theta_0 \{L_{i,j}^{n+m}(k') - L_{i,j}^{n+m}(k)\} \right)}{E_n \exp \left(\theta_0 \{L_{i,j}^{n+m}(k') - L_{i,j}^{n+m}(k)\} \right)}.$$

We also denote

$$\psi_n(m, i, j, k, k') := \log E_n \exp \left(\theta_0 \{L_{i,j}^{n+m}(k') - L_{i,j}^{n+m}(k)\} \right)$$

Observe that

$$\begin{aligned} b_n(m) &= \sum_{0 \leq k < k' \leq 2^{n+m-1}} 6 \exp \left(-\frac{1}{2} \gamma (k' - k)^{\beta-1/2} \Delta_n^{2(\alpha-\alpha')} \Delta_m^{2\alpha-1} \right) \\ &\leq 2^{2(m+n)} \exp \left(-\frac{1}{2} \gamma \Delta_n^{2(\alpha-\alpha')} \Delta_m^{2\alpha-1} \right). \end{aligned}$$

Thus, $b_n(m) \rightarrow 0$ as $n \rightarrow \infty$. Then we can select any probability mass function $\{g(m) : m \geq 1\}$, e.g. $g(m) = e^{-1}/(m-1)!$ for $m \geq 1$, by assuming that n is sufficiently large, such that

$$g(m) \geq d'^2 b_n(m)$$

Now consider the following procedure, which we called Procedure Aux, for "auxiliar", which is given for pedagogical purposes because as we shall see shortly is not directly applicable but just useful to understand the nature of the method that we shall ultimately use.

Procedure Aux

Input: We assume that we have simulated $\{(W_{i,k}^n : 0 \leq k < 2^l) : l \leq n\}$.

Output: A Bernoulli F with parameter $P_n(\tau_1(n) < \infty)$, and if $F = 1$, also

$$\omega_{n:n+M} = \{W_{i,k}^l : 1 \leq k \leq 2^n, 1 \leq i \leq d', n < l \leq n+M\}$$

conditional on the event $\tau_1(n) < \infty$.

Step 1: Sample M according to $g(m)$.

Step 2: Given $M = m$ sample I and J i.i.d. from the uniform distribution over the set $\{1, 2, \dots, d'\}$. Then, sample K', K from $q_n(k, k'|m)$.

Step 3: Given $M = m, I = i, J = j, K = k$, and $K' = k'$, simulate $\omega_{n:n+m}$ from $P_{n,m}^{i,j,k,k'}(\cdot)$. Note that simulation from $P_{n,m}^{i,j,k,k'}(\cdot)$ can be done according to Corollary 3.

Step 4: Compute

$$\Xi_n(m, i, j, k, k', \omega_{n:n+m}) = \frac{1}{g(m) d'^{-2} q_n(k, k'|m) \exp \left(\theta_0 \{L_{i,j}^{n+m}(k') - L_{i,j}^{n+m}(k)\} - \psi_n(m, i, j, k, k') \right)},$$

and

$$\mathcal{N}_n(m) = \sum_{1 \leq i, j \leq d'} \sum_{1 \leq h < h' \leq 2^{n+m-1}} I \left(\left| L_{i,j}^{n+m}(h') - L_{i,j}^{n+m}(h) \right| > (h - h')^\beta \Delta_{n+m}^{2\alpha} \right).$$

Step 5: Simulate U uniformly distributed on $[0, 1]$ independent of everything else and output

$$F = I \left(U < I \left(\left\{ \left| L_{i,j}^{n+m}(k') - L_{i,j}^{n+m}(k) \right| > (k - k')^\beta \Delta_{n+m}^{2\alpha} \right\} \cap \cap_{l=1}^{m-1} \bar{\mathcal{C}}_n(l) \right) \Xi_n(m, i, j, k, k', \omega_{n:n+m}) / \mathcal{N}_n(m) \right).$$

If $F = 1$, also output $\omega_{n:n+m}$.

We claim that the output F is distributed as a Bernoulli random variable with parameter $P_n(\tau_1(n) < \infty)$. Moreover, we claim that if $F = 1$, then, $\omega_{n:n+M}$ is distributed according to $P_n(\omega_{n:n+\tau_1(n)} \in \cdot \mid \tau_1(n) < \infty)$. We first verify the claim that the outcome in Step 5 follows a Bernoulli with parameter $P_n(\tau_1(n) < \infty)$.

In order to see this, let Q_n denote the distribution induced by Procedure Aux. Note that

$$\begin{aligned}
& Q_n(U < I \left(\left\{ \left| L_{i,j}^{n+M}(K') - L_{i,j}^{n+M}(K) \right| > (K - K')^\beta \Delta_{n+M}^{2\alpha} \right\} \cap \cap_{l=1}^{M-1} \bar{\mathcal{C}}_n(l) \right) \\
& \quad \times \Xi_n(M, I, J, K, K', \omega_{n:n+M}) / \mathcal{N}_n(m)) \\
&= E^{Q_n} [I \left(\left\{ \left| L_{i,j}^{n+M}(K') - L_{i,j}^{n+M}(K) \right| > (K - K')^\beta \Delta_{n+M}^{2\alpha} \right\} \cap \cap_{l=1}^{M-1} \bar{\mathcal{C}}_n(l) \right) \\
& \quad \times \Xi_n(M, I, J, K, K', \omega_{n:n+M}) / \mathcal{N}_n(m)] \\
&= \sum_{m=1}^{\infty} \sum_{1 \leq i, j \leq d'} \sum_{1 \leq k < k' \leq 2^{n+m-1}} E^{Q_n} [I \left(\left\{ \left| L_{i,j}^{n+m}(k') - L_{i,j}^{n+m}(k) \right| > (k - k')^\beta \Delta_{n+m}^{2\alpha} \right\} \cap \cap_{l=1}^{m-1} \bar{\mathcal{C}}_n(l) \right) \\
& \quad \times \frac{dP_n}{dP_{n,m}^{i,j,k,k'}}(\omega_{n:n+m}) \times \frac{1}{\mathcal{N}_n(m)}] \\
&= \sum_{m=1}^{\infty} \sum_{1 \leq i, j \leq d'} \sum_{1 \leq k < k' \leq 2^{n+m-1}} E_n \left(\frac{I \left(\left\{ \left| L_{i,j}^{n+m}(k') - L_{i,j}^{n+m}(k) \right| > (k - k')^\beta \Delta_{n+m}^{2\alpha} \right\} \cap \cap_{l=1}^{m-1} \bar{\mathcal{C}}_n(l) \right)}{\mathcal{N}_n(m)} \right) \\
&= \sum_{m=1}^{\infty} P_n(\mathcal{C}_n(m) \cap \cap_{l=1}^{m-1} \bar{\mathcal{C}}_n(l)) = P_n(\tau_1(n) < \infty).
\end{aligned}$$

Similarly,

$$\begin{aligned}
& Q_n(\omega_{n:n+M} \in A \mid U < I(\mathcal{C}_n(M) \cap \cap_{l=1}^{M-1} \bar{\mathcal{C}}_n(l)) \Xi_n(M, I, J, K, K', \omega_{n:n+M})) \\
&= \sum_{m=1}^{\infty} E^{Q_n} \left(\omega_{n:n+m} \in A, \frac{dP_{n,m}^{I,J,K,K'}}{dP_n}(\omega_{n:n+m}) I(\mathcal{C}_n(m) \cap \cap_{l=1}^{m-1} \bar{\mathcal{C}}_n(l)) \right) / P_n(\tau_1(n) < \infty) \\
&= \sum_{m=1}^{\infty} P_n(\omega_{n:n+m} \in A, \tau_1(n) = m) / P_n(\tau_1(n) < \infty) \\
&= P_n(\omega_{n:n+\tau_1(n)} \in A \mid \tau_1(n) < \infty)
\end{aligned}$$

The deficiency of Procedure Aux is that it does not recognize that $n > N_1$. Let us now account for this fact and note that conditional on \mathcal{F}_{N_1} we have that $W_{i,k}^n$'s are i.i.d. $N(0, 1)$ but conditional on $\{|W_{i,k}^n| < 4\sqrt{n+1}\}$ for all $n > N_1$. Define

$$\mathcal{H}_m^n = \{|W_{i,k}^r| < 4\sqrt{r+1} : 1 \leq k \leq 2^r, n < r \leq n+m\}.$$

In order to simulate $P_{N_1}(\tau_1(N_1) < \infty)$ we modify step 3 of Procedure Aux. Specifically, we have

Procedure B

Input: We assume that we have simulated $\{(W_{i,k}^l : 0 \leq k < 2^l) : l \leq n\}$. So, the $W_{i,k}^m$'s are i.i.d. $N(0, 1)$ but conditional on $\{|W_{i,k}^m| < 4\sqrt{m+1}\}$ for all $m > n$. We also assume that conditions (25) and (26) hold in Lemma 7; note the discussion following Lemma 7 which notes that this can be assumed at the expense of simulating additional $W_{i,k}^m$'s (with $\{|W_{i,k}^m| < 4\sqrt{m+1}\}$ if $m > N_1$).

Output: A Bernoulli F with parameter $P_n(\tau_1(n) < \infty, \mathcal{H}_\infty^n)$, and if $F = 1$, also

$$\omega_{n:n+\tau_1(n)} = \{W_{i,k}^l : 1 \leq k \leq 2^n, 1 \leq i \leq d', n < l \leq n + \tau_1(n)\}$$

conditional on $\tau_1(n) < \infty$ and on \mathcal{H}_∞^n .

Step 1: Sample M according to $g(m)$.

Step 2: Given $M = m$ sample I and J i.i.d. from the uniform distribution over the set $\{1, 2, \dots, d'\}$. Then, sample K', K from $q_n(k, k'|m)$.

Step 3: Given $M = m$, $I = i, J = j, K = k$, and $K' = k'$, simulate $\omega_{n:n+m}$ from $P_{n,m}^{i,j,k,k'}(\cdot)$. Note that simulation from $P_{n,m}^{i,j,k,k'}(\cdot)$ can be done according to Corollary 3.

Step 4: Compute

$$\Xi_n(m, i, j, k, k', \omega_{n:n+m}) = \frac{1}{g(m)d'^{-2}q_n(k, k'|m) \exp\left(\theta_0\{L_{i,j}^{n+m}(k') - L_{i,j}^{n+m}(k)\} - \psi_n(m, i, j, k, k')\right)},$$

and

$$\mathcal{N}_n(m) = \sum_{1 \leq i, j \leq d'} \sum_{1 \leq k < k' \leq 2^{n+m-1}} I\left(\left|L_{i,j}^{n+m}(k') - L_{i,j}^{n+m}(k)\right| > (k - k')^\beta \Delta_{n+m}^{2\alpha}\right).$$

Step 5: Simulate U uniformly distributed on $[0, 1]$ independent of everything else and output

$$F = I(U < \frac{I\left(\mathcal{H}_m^n \cap \left\{\left|L_{i,j}^{n+m}(k') - L_{i,j}^{n+m}(k)\right| > (k - k')^\beta \Delta_{n+m}^{2\alpha}\right\} \cap \cap_{l=1}^{M-1} \bar{\mathcal{C}}_n(l)\right) P(\mathcal{H}_\infty^{n+m})}{P(\mathcal{H}_\infty^n)}) \\ \times \Xi_n(m, i, j, k, k', \omega_{n:n+m}) / \mathcal{N}_n(m))$$

(Notice that $P(\mathcal{H}_\infty^{n+m})/P(\mathcal{H}_\infty^n) = P(\mathcal{H}_{n+m}^n)$ and can be computed in finite steps.)

If $F = 1$, also output $\omega_{n:n+m}$.

Let \tilde{Q}_n denote the distribution induced by Procedure \tilde{B} . Following the same analysis as that given for Procedure B, we can verify that

$$\tilde{Q}_n(U < \frac{I\left(\mathcal{H}_m^n \cap \left\{\left|L_{i,j}^{n+m}(k') - L_{i,j}^{n+m}(k)\right| > (k - k')^\beta \Delta_{n+m}^{2\alpha}\right\} \cap \cap_{l=1}^{M-1} \bar{\mathcal{C}}_n(l)\right) P(\mathcal{H}_\infty^{n+m})}{P(\mathcal{H}_\infty^n)}) \\ \times \Xi_n(m, i, j, k, k', \omega_{n:n+m}) / \mathcal{N}_n(m)) = P_n(\tau_1(n) < \infty | \mathcal{H}_\infty^n).$$

And if the Bernoulli trial is a success, then, $\omega_{n:n+M}$ is distributed according to

$$P_n(\omega_{n:n+\tau_1(n)} \in \cdot \mid \tau_1(n) < \infty, \mathcal{H}_\infty^n).$$

Finally, if $\tau_1(n) = \infty$, we still need to simulate $\omega_{n:n+m}$ for any $m \geq 1$. But now, conditional on $\{\tau_1(n) = \infty, \mathcal{H}_\infty^n\}$. Note that

$$P_n(\omega_{n:n+m} \in A \mid \tau_1(n) = \infty, \mathcal{H}_\infty^n) = \frac{P_n(\omega_{n:n+m} \in A, \tau_1(n) = \infty, \mathcal{H}_\infty^n)}{P_n(\tau_1(n) = \infty, \mathcal{H}_\infty^n)} \\ = \frac{E_n I(\omega_{n:n+m} \in A, \tau_1(n) > m, \mathcal{H}_m^n) P_{n+m}(\tau_1(n+m) = \infty, \mathcal{H}_\infty^{n+m})}{P_n(\tau_1(n) = \infty, \mathcal{H}_\infty^n)}.$$

We do this by sampling $\omega_{n:n+m}$ from $P_n(\cdot)$ and accept the path with probability

$$I(\tau_1(n) > m, \mathcal{H}_m^n) P_{n+m}(\tau_1(n+m) = \infty, \mathcal{H}_\infty^{n+m}).$$

This clearly can be done since we can easily simulate Bernoullis with probability

$$P_{n+m}(\tau(n+m) = \infty, \mathcal{H}_\infty^{n+m}) = P_{n+m}(\tau_1(n+m) = \infty \mid \mathcal{H}_\infty^{n+m}) P_{n+m}(\mathcal{H}_\infty^{n+m}).$$

We summarize the algorithm as follows:

Algorithm II: Simulate N_1 and N_2 jointly with $W_{i,k}^n$'s for $1 \leq n \leq N_0$, where N_0 is chosen such that $\sup_{t \in [0,1]} \|\hat{X}^{N_0}(t) - X(t)\|_\infty \leq \varepsilon$

Input: The parameters required to run Algorithm I, and Procedures A and B. These are the tilting parameters θ_0 's.

Step 1: Simulate N_1 jointly with $W_{i,k}^m$'s for $0 \leq m \leq N_1$ using Algorithm I (see the remark that follows after Algorithm I). Let $n = N_1$.

Step 2: If any of the conditions (25) and (26) from Lemma 7 are not satisfied keep simulating $W_{i,k}^m$'s for $m > n$ until the first level $m > n$ for which conditions (25) and (26) are satisfied. Redefine n to be such first level m .

Step 3: Run Procedure B and obtain as output F and if $F = 1$ also obtain $\omega_{n:n+\tau(n)}$.

Step 4: If $\tau(n) < \infty$ (i.e. $F = 1$) set $n \leftarrow \tau(n)$ and go back to Step 2. Otherwise, go to Step 4.

Step 5: Calculate G according to Procedure A and solve for N_0 such that $G\Delta_{N_0}^{2\alpha-\beta} < \varepsilon$.

Step 6: If $N_0 > n$ sample $\omega_{n:N_0}$ from $P_n(\cdot)$ and sample a Bernoulli random variable, I with probability of success $P_{N_0}(\tau(N_0) = \infty, \mathcal{H}_\infty^{N_0})$.

Step 7: If $I = 0$, go back to Step 6.

Step 8: Output $\omega_{0:N_0}$.

We obtain $\{W_{i,k}^l : 0 \leq k < 2^l, l \leq N_0, 1 \leq i \leq d\}$ from Algorithm II. We have from recursions (17) and (18) how to obtain

$$(29) \quad \{(Z_i(t_r^l) - Z_i(t_{r-1}^l)) : 1 \leq r \leq 2^l, 1 \leq l \leq N_0, 1 \leq i \leq d\}$$

and then we can compute $\{\hat{X}^{N_0}(t) : t \in D_{N_0}\}$ using equation (7).

Remark: Observe that after completion of Algorithm II, one can actually continue the simulation of increments in order to obtain an approximation with an error $\varepsilon' < \varepsilon$. In particular, this is done by repeating Steps 4 to 8. Start from Step 4 with $n = N_0$. The value of G has been computed, it does not depend on ε . However, one needs to recompute $N_0 := N_0(\varepsilon')$ such that $G\Delta_{N_0}^{2\alpha-\beta} < \varepsilon'$. Then we can implement Steps 5 to 8 without change. One obtains an output that, as before, can be transformed into (29) via the recursions (17), yielding $\{\hat{X}^{N_0(\varepsilon')}(t) : t \in D_{N_0(\varepsilon')}\}$ with a guaranteed error smaller than ε' in uniform norm with probability 1.

4. ROUGH PATH DIFFERENTIAL EQUATIONS, ERROR ANALYSIS, AND THE PROOF OF THEOREM 1

The analysis in this section follows closely the discussion from [7] Section 3 and Section 7; see also [8] Chapter 10. We made some modifications to account for the drift of the process and also to be able to explicitly calculate the constant G . Let us start with the definition of a solution to (1) using the theory of rough differential equations.

Definition 1. $X(\cdot)$ is a solution of (1) on $[0, 1]$ if $X(0) = x(0)$ and

$$\begin{aligned} & |X_i(t) - X_i(s) - \mu_i(X(s))(t-s) - \sum_{j=1}^{d'} \sigma_{i,j}(X(s))(Z_j(t) - Z_j(s)) \\ & - \sum_{j=1}^{d'} \sum_{l=1}^d \sum_{m=1}^{d'} \partial_l \sigma_{i,j}(X(s)) \sigma_{l,m}(X(s)) A_{m,j}(s, t)| = o(t-s) \end{aligned}$$

for all i and $0 \leq s < t \leq 1$, where $A_{i,j}(\cdot)$ satisfies

$$(30) \quad A_{i,j}(r, t) = A_{i,j}(r, s) + A_{i,j}(s, t) + (Z_i(s) - Z_i(r))(Z_j(t) - Z_j(s))$$

for $0 \leq r < s < t \leq 1$.

The previous definition is motivated by the following Taylor-type development,

$$\begin{aligned}
X_i(t+h) &= X_i(t) + \int_t^{t+h} \mu_i(X(u))du + \sum_{j=1}^{d'} \int_t^{t+h} \sigma_{i,j}(X(u))dZ_j(u) \\
&\approx X_i(t) + \int_t^{t+h} \mu_i(X(u))du \\
&\quad + \sum_{j=1}^{d'} \int_t^{t+h} \sigma_{i,j}(X(t) + \mu(X(t))(u-t) + \sigma(X(t))(Z(u) - Z(t)))dZ_j(u) \\
&\approx X_i(t) + \mu_i(X(t))h + \sum_{j=1}^{d'} \sigma_{i,j}(X(t))(Z_j(t+h) - Z_j(t)) \\
&\quad + \sum_{j=1}^{d'} \sum_{l=1}^d \sum_{m=1}^{d'} \partial_l \sigma_{i,j}(X(t)) \sigma_{l,m}(X(t)) \int_t^{t+h} (Z_m(u) - Z_m(t))dZ_j(u).
\end{aligned}$$

The previous Taylor development suggests defining $A_{i,j}(s, t) := \int_s^t (Z_i(u) - Z_i(s))dZ_j(u)$. Depending on how one interprets $A(s, t)$, e.g. via Ito or Stratonovich integrals, one obtains a solution $X(\cdot)$ which is interpreted in the corresponding context.

In order to obtain the Ito interpretation of the solution to equation (1) via definition (1) we shall interpret the integrals in the sense of Ito. In addition, as we shall explain, some technical conditions (in addition to the standard Lipschitz continuity typically required to obtain a strong solution a la Ito) must be imposed in order to enforce the existence of a unique solution to (1).

There are two sources of errors when using \hat{X}^n in equation (7) to approximate X . One is the discretization on the dyadic grid, but assuming that $A_{i,j}(t_k^n, t_{k+1}^n)$ is known; this type of analysis is the one that is most common in the literature on rough paths (see [7]). The second source of error arises precisely accounting for the fact that $A_{i,j}(t_k^n, t_{k+1}^n)$ is not known. Thus we divide the proof of Theorem 1 into two steps (two propositions), each dealing with one source of error.

Similar to $\hat{X}^n(t)$, we define $\{X^n(t) : t \in D_n\}$ by the following recursion: given $X^n(0) = X(0)$,

$$\begin{aligned}
(31) \quad X_i^n(t_{k+1}^n) &= X_i^n(t_k^n) + \mu_i(X^n(t_k^n))\Delta_n + \sum_{j=1}^{d'} \sigma_{i,j}(X_i^n(t_k^n))(Z_j(t_{k+1}^n) - Z_j(t_k^n)) \\
&\quad + \sum_{j=1}^{d'} \sum_{l=1}^d \sum_{m=1}^{d'} \partial_l \sigma_{i,j}(X_i^n(t_k^n)) \sigma_{l,m}(X_i^n(t_k^n)) A_{m,j}(t_k^n, t_{k+1}^n),
\end{aligned}$$

and for $t \in [0, 1]$, we let $X^n(t) = X^n(\lfloor t \rfloor)$, where in this context $\lfloor t \rfloor = \max\{s \in D_n : s \leq t\}$.

Proposition 2. *Under the conditions of Theorem 1, we can compute a constant G_1 explicitly in terms of M , $\|Z\|_\alpha$ and $\|A\|_{2\alpha}$, such that for n large enough*

$$\|X^n(t) - X(t)\|_\infty \leq G_1 \Delta_n^{3\alpha-1}.$$

The proof of Proposition 2 will be given after introducing some definitions and key auxiliary results. We denote

$$I_i^n(r, t) := X_i^n(t) - X_i^n(r) - \mu_i(X^n(r))(t - r) - \sum_{j=1}^{d'} \sigma_{i,j}(X^n(r))(Z_j(t) - Z_j(r))$$

and

$$J_i^n(r, t) := I_i^n(r, t) - \sum_{j=1}^{d'} \sum_{l=1}^d \sum_{m=1}^{d'} \partial_l \sigma_{i,j}(X^n(r)) \sigma_{l,m}(X^n(r)) A_{m,j}(r, t).$$

The following lemmas introduce the main technical results for the proof of Proposition 2.

Lemma 8. *Under the conditions of Theorem 1, there exist constants C_1 , C_2 and C_3 that depend only on M , $\|Z\|_\alpha$ and $\|A\|_{2\alpha}$, such that for any large enough n and $r, t \in D_n$,*

$$\begin{aligned} \|X^n(t) - X^n(r)\|_\infty &\leq C_1 |t - r|^\alpha, \\ |I^n(r, t)|_\infty &\leq C_2 |t - r|^{2\alpha}, \end{aligned}$$

and

$$\|J^n(r, t)\|_\infty \leq C_3 |t - r|^{3\alpha}.$$

Proof. We follow For $r \leq s \leq t$, $r, s, t \in D_n$, we have the following important recursions:

$$I_i^n(r, t) = I_i^n(r, s) + I_i^n(s, t) + (\mu_i(X^n(s)) - \mu_i(X^n(r)))(t - s) + \sum_{j=1}^{d'} (\sigma_{i,j}(X^n(s)) - \sigma_{i,j}(X^n(r)))(Z_j(t) - Z_j(s))$$

and

$$\begin{aligned} &J_i^n(r, t) \\ &= J_i^n(r, s) + J_i^n(s, t) + (\mu_i(X^n(s)) - \mu_i(X^n(r)))(t - s) \\ &\quad + \sum_{j=1}^{d'} [\sigma_{i,j}(X^n(s)) - \sigma_{i,j}(X^n(r)) - \sum_{l=1}^d \partial_l \sigma_{i,j}(X^n(r))(X_l^n(s) - X_l^n(r)) \\ &\quad + \sum_{l=1}^d \partial_l \sigma_{i,j}(X^n(r)) I_l^n(r, s)] (Z_j(t) - Z_j(s)) \\ (32) \quad &+ \sum_{j=1}^{d'} \sum_{l=1}^d \sum_{m=1}^{d'} [\partial_l \sigma_{i,j}(X^n(s)) \sigma_{l,m}(X^n(s)) - \partial_l \sigma_{i,j}(X^n(r)) \sigma_{l,m}(X^n(r))] A_{m,j}(s, t) \end{aligned}$$

We next divide the proof into two parts. We first prove that there exists a small enough constant $\delta > 0$ and three large enough constants $C_1(\delta)$, $C_2(\delta)$ and $C_3(\delta)$, all independent of n , such that for $|t - r| < \delta$, $\|X^n(t) - X^n(r)\|_\infty \leq C_1(\delta) |t - r|^\alpha$, $\|I^n(r, t)\|_\infty \leq C_2(\delta) |t - r|^{2\alpha}$ and $\|J^n(r, t)\|_\infty \leq C_3(\delta) |t - r|^{3\alpha}$. We prove it by induction. First we have $J^n(r, r) = 0$ and $J^n(r, r + \Delta_n) = 0$. Suppose the result hold for all pairs of $r_0, t_0 \in D_n$ with $|t_0 - r_0| < |t - r|$. We then pick $s \in D_n$ as the largest point between r and t such that $|s - r| \leq |t - r|/2$. Then we also have $|s + \Delta_n - r| > |t - r|/2$ and $|t - (s + \Delta_n)| < |t - r|/2$.

As

$$\begin{aligned} X_i^n(t) - X_i^n(s) &= J_i^n(s, t) + \mu_i(X^n(s))(t - s) + \sum_{j=1}^{d'} \sigma_{i,j}(X^n(s))(Z_j(t) - Z_j(s)) \\ &\quad + \sum_{j=1}^{d'} \sum_{l=1}^d \sum_{m=1}^{d'} \partial_l \sigma_{i,j}(X^n(s)) \sigma_{l,m}(X^n(s)) A_{m,j}(s, t), \end{aligned}$$

we have

$$\begin{aligned} |X_i^n(t) - X_i^n(s)| &\leq C_3(\delta)|t-s|^{3\alpha} + M|t-s| + dM\|Z\|_\alpha|t-s|^\alpha + d^3M^2\|A\|_{2\alpha}|t-s|^{2\alpha} \\ &\leq (C_3(\delta)\delta^{2\alpha} + M\delta^{1-\alpha} + dM\|Z\|_\alpha + d^3M^2\|A\|_{2\alpha}\delta^\alpha)|t-s|^\alpha \\ &\leq C_1(\delta)|t-s|^\alpha \end{aligned}$$

for $C_1(\delta) > C_3(\delta)\delta^{2\alpha} + M\delta^{1-\alpha} + dM\|Z\|_\alpha + d^3M^2\|A\|_{2\alpha}\delta^\alpha$.

And as

$$I_i^n(s, t) = J_i^n(s, t) + \sum_{j=1}^{d'} \sum_{l=1}^d \sum_{m=1}^{d'} \partial_l \sigma_{i,j}(X^n(s)) \sigma_{l,m}(X^n(s)) A_{m,j}(s, t),$$

we have

$$|I_i^n(s, t)| \leq C_3(\delta)|t-s|^{3\alpha} + d^3M^2\|A\|_{2\alpha}|t-s|^{2\alpha} \leq (C_3(\delta)\delta^\alpha + d^3M^2\|A\|_{2\alpha})|t-s|^{2\alpha} \leq C_2(\delta)|t-s|^{2\alpha}$$

for $C_2(\delta) > C_3(\delta)\delta^\alpha + d^3M^2\|A\|_{2\alpha}$.

We now analyze the recursion (32) term by term. First,

$$|\mu_i(X^n(s)) - \mu_i(X^n(r))| \leq MC_1(\delta)|s-r|^\alpha,$$

$$|\sigma_{i,j}(X^n(s)) - \sigma_{i,j}(X^n(r)) - \sum_{l=1}^d \partial_l \sigma_{i,j}(X^n(r))(X_l^n(s) - X_l^n(r))| \leq MC_1(\delta)^2|s-r|^{2\alpha},$$

$$|\sum_{l=1}^d \partial_l \sigma_{i,j}(X^n(r)) I_l^n(r, s)| \leq dMC_2(\delta)|s-r|^{2\alpha},$$

and

$$|\partial_l \sigma_{i,j}(X^n(s)) \sigma_{l,m}(X^n(s)) - \partial_l \sigma_{i,j}(X^n(r)) \sigma_{l,m}(X^n(r))| \leq 2M^2C_1(\delta)|s-r|^\alpha.$$

Then

$$\begin{aligned} |J_i^n(r, t)| &\leq |J_i^n(r, s)| + |J_i^n(s, t)| \\ &\quad + (MC_1(\delta) + dMC_1(\delta)^2\|Z\|_\alpha + d^2MC_2(\delta)\|Z\|_\alpha + 2d^3M^2C_1(\delta)\|A\|_\alpha)|t-r|^{3\alpha} \end{aligned}$$

Likewise, we have

$$\begin{aligned} |J_i^n(s, t)| &\leq |J_i^n(s, s + \Delta_n)| + |J_i^n(s + \Delta_n, t)| \\ &\quad + (MC_1(\delta) + dMC_1(\delta)^2\|Z\|_\alpha + d^2MC_2(\delta)\|Z\|_\alpha + 2d^3M^2C_1(\delta)\|A\|_\alpha)|t-s|^{3\alpha} \\ &= |J_i^n(s + \Delta_n, t)| \\ &\quad + (MC_1(\delta) + dMC_1(\delta)^2\|Z\|_\alpha + d^2MC_2(\delta)\|Z\|_\alpha + 2d^3M^2C_1(\delta)\|A\|_\alpha)|t-s|^{3\alpha}. \end{aligned}$$

Then

$$\begin{aligned} |J_i^n(r, t)| &\leq |J_i^n(r, s)| + |J_i^n(s + \Delta_n, t)| \\ &\quad + 2(MC_1(\delta) + dMC_1(\delta)^2\|Z\|_\alpha + d^2MC_2(\delta)\|Z\|_\alpha + 2d^3M^2C_1(\delta)\|A\|_\alpha)|t-s|^{3\alpha} \\ &\leq (2^{1-3\alpha}C_3(\delta) + 2(MC_1(\delta) + dMC_1(\delta)^2\|Z\|_\alpha + d^2MC_2(\delta)\|Z\|_\alpha + 2d^3M^2C_1(\delta)\|A\|_\alpha))|t-s|^{3\alpha} \\ &\leq C_3(\delta)|t-s|^{3\alpha}, \end{aligned}$$

for

$$(1 - 2^{1-3\alpha})C_3(\delta) > 2(MC_1(\delta) + dMC_1(\delta)^2\|Z\|_\alpha + d^2MC_2(\delta)\|Z\|_\alpha + 2d^3M^2C_1(\delta)\|A\|_\alpha).$$

Therefore, if we deliberately choose δ , $C_1(\delta)$, $C_2(\delta)$ and $C_3(\delta)$ such that

$$\begin{aligned} C_1(\delta) &> C_3(\delta)\delta^{2\alpha} + M\delta^{1-\alpha} + dM\|Z\|_\alpha + d^3M^2\|A\|_{2\alpha}\delta^\alpha \\ C_2(\delta) &> C_3(\delta)\delta^\alpha + d^3M^2\|A\|_{2\alpha} \\ C_3(\delta) &> \frac{2}{1-2^{1-3\alpha}} (MC_1(\delta) + dMC_1(\delta)^2\|Z\|_\alpha + d^2MC_2(\delta)\|Z\|_\alpha + 2d^3M^2C_1(\delta)\|A\|_\alpha) \end{aligned}$$

Then we have

$$\begin{aligned} \|X^n(t) - X^n(r)\|_\infty &\leq C_1(\delta)|t-r|^\alpha, \\ \|I^n(r, t)\|_\infty &\leq C_2(\delta)|t-r|^{2\alpha}, \\ \|J^n(r, t)\|_\infty &\leq C_3(\delta)|t-r|^{3\alpha}, \end{aligned}$$

for $|t-r| < \delta$.

We now extend the analysis to the case when $|t-r| > \delta$. For n large enough ($\Delta_n < \delta/2$), if $|t-r| > \delta$, we can always find points $s_i \in D_n$ and $r = s_0 < s_1 < \dots < s_k = t$ such that $\max_{1 \leq i \leq k} |s_i - s_{i-1}| < \delta$ and $\min_{1 \leq i \leq k} |s_i - s_{i-1}| > \delta/2$. Then

$$|X_i^n(t) - X_i^n(r)| \leq \sum_{l=1}^k |X_i^n(s_l) - X_i^n(s_{l-1})| \leq kC_1(\delta)|t-r|^\alpha \leq \frac{2}{\delta}C_1(\delta)|t-r|^\alpha$$

Let $C_1 = \frac{2}{\delta}C_1(\delta)$ and we can write $\|X^n(t) - X^n(r)\|_\infty < C_1|t-r|^\alpha$. Next,

$$\begin{aligned} |I_i^n(r, t)| &\leq \sum_{l=1}^k \{ |I_i^n(s_{l-1}, s_l)| + |(\mu_i(X^n(s_l)) - \mu_i(X^n(s_0)))(s_l - s_{l-1})| \\ &\quad + |(\sigma_i(X^n(s_l)) - \sigma_i(X^n(s_0)))(Z(s_{l+1}) - Z(s_l))| \} \\ &\leq k[C_2(\delta)|t-r|^{2\alpha} + MC_1|t-r|^{1+\alpha} + dMC_1\|Z\|_\alpha|t-r|^{2\alpha}] \\ &\leq \frac{2}{\delta}(C_2(\delta) + MC_1 + dMC_1\|Z\|_\alpha)|t-r|^{2\alpha} \end{aligned}$$

By setting $C_2 = \frac{2}{\delta}(C_2(\delta) + MC_1 + dMC_1\|Z\|_\alpha)$, we have $\|I^n(r, t)\|_\infty < C_2|t-r|^{2\alpha}$.

Now following the same induction analysis on $J_i^n(s, t)$ as we did in the case $|t-s| < \delta$, we have

$$|J_i^n(r, t)| \leq \frac{2}{2^{3\alpha}}C_3|t-r|^{3\alpha} + 2(MC_1 + dMC_1^2\|Z\|_\alpha + d^2MC_2\|Z\|_\alpha + 2d^3M^2C_1\|A\|_\alpha)|t-r|^{3\alpha}$$

If we choose

$$C_3 = \frac{2}{1-2^{1-3\alpha}}(MC_1 + dMC_1^2\|Z\|_\alpha + d^2MC_2\|Z\|_\alpha + 2d^3M^2C_1\|A\|_\alpha),$$

then $\|J^n(r, t)\|_\infty \leq C_3|t-s|^{3\alpha}$.

□

Lemma 9. Let $x(0)$ and $\tilde{x}(0) \in \mathbb{R}^d$ be two different vectors. We denote $X^n(t)$ and $\tilde{X}^n(t)$ for $t \in D_n$ as the n -th dyadic approximation defined by (31) with initial value $x(0)$ and $\tilde{x}(0)$ respectively. Under the conditions of Theorem 1, there exists a constant B , independent of n , such that for $t \in D_n$,

$$\|X^n(t) - \tilde{X}^n(t) - (X^n(0) - \tilde{X}^n(0))\|_\infty \leq Bt^\alpha\|X^n(0) - \tilde{X}^n(0)\|_\infty.$$

Moreover,

$$\|X^n(t) - \tilde{X}^n(t)\|_\infty \leq (1+B)\|X^n(0) - \tilde{X}^n(0)\|_\infty.$$

Proof. Let

$$Y_{i,h}^n(t) = \frac{X_i^n(t) - \tilde{X}_i^n(t)}{\|X_h^n(0) - \tilde{X}_h^n(0)\|_\infty}$$

We define $0/0 = 0$.

Then following the recursion (31), we have

$$\begin{aligned} & Y_{i,h}^n(t_{k+1}^n) \\ &= Y_{i,h}^n(t_k^n) + \frac{\mu_i(X^n(t_k^n)) - \mu_i(\tilde{X}^n(t_k^n))}{\|X_h^n(0) - \tilde{X}_h^n(0)\|_\infty} \Delta_n + \sum_{j=1}^{d'} \frac{\sigma_{i,j}(X^n(t_k^n)) - \sigma_{i,j}(\tilde{X}^n(t_k^n))}{\|X_h^n(0) - \tilde{X}_h^n(0)\|_\infty} (Z_j(t_{k+1}^n) - Z_j(t_k^n)) \\ (33) \quad & + \sum_{j=1}^{d'} \sum_{l=1}^d \sum_{m=1}^{d'} \frac{\partial_l \sigma_{i,j}(X^n(t_k^n)) \sigma_{l,m}(X^n(t_k^n)) - \partial_l \sigma_{i,j}(\tilde{X}^n(t_k^n)) \sigma_{l,m}(\tilde{X}^n(t_k^n))}{\|X_h^n(0) - \tilde{X}_h^n(0)\|_\infty} A_{m,j}(t_k^n, t_{k+1}^n) \end{aligned}$$

Then (31) and (33) together define an recursion to generate X^n , \tilde{X}^n and Y^n . Following Lemma 8, there exists a constant B that depends only on M , $\|Z\|_\alpha$ and $\|A\|_{2\alpha}$, such that

$$\|Y^n(t) - Y^n(0)\|_\infty \leq Bt^\alpha.$$

Thus,

$$\|X^n(t) - \tilde{X}^n(t) - (X^n(0) - \tilde{X}^n(0))\|_\infty \leq Bt^\alpha \|X^n(0) - \tilde{X}^n(0)\|_\infty,$$

and

$$\|X^n(t) - \tilde{X}^n(t)\|_\infty \leq (1 + B) \|X^n(0) - \tilde{X}^n(0)\|_\infty.$$

□

We are now ready to prove Proposition 2.

Proof of Proposition 2. From Lemma 8 we have $\|X^n(t) - X^n(r)\|_\infty \leq C_1|t - r|^\alpha$. By Arzela-Ascoli Theorem, there exists a subsequence of $\{X^n\}$ that converges uniformly to some continuous function X on $[0, 1]$. Moreover we have $\|X(t) - X(r)\|_\infty \leq C_1|t - r|^\alpha$ and

$$\begin{aligned} & |X_i(t) - X_i(r) - \mu_i(X(r)) - \sum_{j=1}^{d'} \sigma_{i,j}(X(r))(Z_j(t) - Z_j(r)) \\ & - \sum_{j=1}^{d'} \sum_{l=1}^d \sum_{m=1}^{d'} \partial_l \sigma_{i,j}(X(r)) \sigma_{l,m}(X(r)) A_{m,j}(r, t)| < C_2|t - r|^{3\alpha} \end{aligned}$$

Therefore, the limit X is a solution to the SDE.

Let $X^{n,(s)}(t; X(s)) := X^n(t - s) | X^n(0) = X(s)$. Specifically, we have $X^{n,(0)}(t; X(0)) = X^n(t)$ with $X^n(0) = X(0)$, and $X^{n,(t)}(t; X(t)) = X(t)$. Then we can write

$$X^n(t_m^n) - X(t_m^n) = \sum_{k=1}^m \left(X^{n,(t_k^n)}(t_m^n; X(t_k^n)) - X^{n,(t_{k-1}^n)}(t_m^n; X(t_{k-1}^n)) \right)$$

By Lemma 9, $\|X^{n,(t_k^n)}(t_m; X(t_k^n)) - X^{n,(t_{k-1}^n)}(t_m; X(t_{k-1}^n))\|_\infty \leq (1+B)\|X(t_k^n) - X^{n,t_{k-1}^n}(t_k^n; X(t_{k-1}^n))\|_\infty$. We also have

$$\begin{aligned} & |X_i(t_k^n) - X_i^{n,(t_{k-1}^n)}(t_k^n; X(t_{k-1}^n))| \\ &= |X_i(t_k^n) - X_i(t_{k-1}^n) - \mu_i(X(t_{k-1}^n))(t_k^n - t_{k-1}^n) - \sum_{j=1}^{d'} \sigma_{i,j}(X(t_{k-1}^n))(Z_j(t_k^n) - Z_j(t_{k-1}^n)) \\ &\quad - \sum_{j=1}^{d'} \sum_{l=1}^d \sum_{m=1}^{d'} \partial_l \sigma_{i,j}(X(t_{k-1}^n)) \sigma_{l,m}(X(t_{k-1}^n)) A_{m,j}(t_{k-1}^n, t_k^n)| \\ &\leq C_3 |t_k^n - t_{k-1}^n|^{3\alpha} \end{aligned}$$

Thus,

$$\begin{aligned} \|X^n(t_m^n) - X(t_m^n)\|_\infty &\leq \sum_{k=1}^m \|X^{n,(t_k^n)}(t_m^n; X(t_k^n)) - X^{n,(t_{k-1}^n)}(t_m^n; X(t_{k-1}^n))\|_\infty \\ &\leq m(1+B)C_3 \Delta_n^{3\alpha} \\ &\leq (1+B)C_3 \Delta_n^{3\alpha-1}. \end{aligned}$$

□

Next we turn to the analysis of the error induced by approximating the Lévy area.

Proposition 3. *Under the conditions of Theorem 1, we can compute a constant G_2 explicitly in terms of M , $\|Z\|_\alpha$, $\|A\|_{2\alpha}$ and Γ_R , such that for n large enough*

$$\|\hat{X}^n(t) - X^n(t)\|_\infty \leq G_2 \Delta_n^{2\alpha-\beta}.$$

The proof of Proposition 3 uses a similar technique as the proof of Proposition 2 and also relies on some auxiliary results. Let

$$U_i^n(s, t) := \hat{X}_i^n(t) - X_i^{n,(s)}(t; \hat{X}^n(s)) + \sum_{j=1}^{d'} \sum_{l=1}^d \sum_{m=1}^{d'} \partial_l \sigma_{i,j}(\hat{X}^n(s)) \sigma_{l,m}(\hat{X}^n(s)) R_{m,j}^n(s, t).$$

We first prove the following technical result.

Lemma 10. *Under the conditions of Theorem 1, there exists a constant C_4 , that depends only on M , $\|Z\|_\alpha$, $\|A\|_{2\alpha}$ and Γ_R , such that*

$$\|U^n(r, t)\|_\infty \leq C_4 |t - r|^{\alpha+\beta} \Delta_n^{2\alpha-\beta}$$

Proof. For $0 \leq r < s < t \leq 1$, $r, s, t \in D_n$, we have

$$\begin{aligned} U_i^n(r, t) &= U_i^n(r, s) + U_i^n(s, t) \\ &\quad + \left[X_i^{n,(s)}(t; \hat{X}^n(s)) - X_i^{n,(r)}(t; \hat{X}^n(r)) - (\hat{X}_i^n(s) - X_i^{n,(r)}(s; \hat{X}^n(r))) \right] \\ &\quad - \sum_{j=1}^{d'} \sum_{l=1}^d \sum_{m=1}^{d'} \left(\partial_l \sigma_{i,j}(\hat{X}^n(s)) \sigma_{l,m}(\hat{X}^n(s)) - \partial_l \sigma_{i,j}(\hat{X}^n(r)) \sigma_{l,m}(\hat{X}^n(r)) \right) R_{m,j}^n(s, t) \end{aligned}$$

From Lemma 9,

$$\begin{aligned} & |X_i^{n,(s)}(t; \hat{X}^n(s)) - X_i^{n,(r)}(t; \hat{X}^n(r)) - (\hat{X}_i^n(s) - X_i^{n,(r)}(s; \hat{X}^n(r)))| \\ &\leq B |t - s|^\alpha \|\hat{X}^n(s) - X^{n,(r)}(s; \hat{X}^n(r))\|_\infty \end{aligned}$$

From Lemma 8,

$$\begin{aligned} & \left| \left(\partial_l \sigma_{i,j}(\hat{X}^n(s)) \sigma_{l,m}(\hat{X}^n(s)) - \partial_l \sigma_{i,j}(\hat{X}^n(r)) \sigma_{l,m}(\hat{X}^n(r)) \right) R_{m,j}^n(s, t) \right| \\ & \leq 2M^2 C_1 |s - r|^\alpha \Gamma_R |t - s|^\beta \Delta_n^{2\alpha-\beta} \\ & \leq 2M^2 C_1 \Gamma_R |t - r|^{\alpha+\beta} \Delta_n^{2\alpha-\beta} \end{aligned}$$

Therefore,

$$\begin{aligned} & \|U^n(r, t)\|_\infty \\ & \leq \|U^n(r, s)\|_\infty + \|U^n(s, t)\|_\infty + B|t - s|^\alpha \|\hat{X}^n(s) - X^{n,(r)}(t_p^n; \hat{X}^n(r))\|_\infty \\ & \quad + 2d^3 M^2 C_1 \Gamma_R |t - r|^{\alpha+\beta} \Delta_n^{2\alpha-\beta} \\ & \leq \|U^n(r, s)\|_\infty + \|U^n(s, t)\|_\infty + B|t - s|^\alpha \|U^n(r, s)\|_\infty \\ & \quad + B|t - s|^\alpha \max_i \left\{ \sum_{j=1}^{d'} \sum_{l=1}^d \sum_{m=1}^{d'} \partial_l \sigma_{i,j}(\hat{X}^n(r)) \sigma_{l,m}(\hat{X}^n(r)) R_{m,j}^n(r, s) \right\} \\ & \quad + 2d^3 M^2 C_1 \Gamma_R |t - r|^{\alpha+\beta} \Delta_n^{2\alpha-\beta} \\ & \leq (1 + B|t - s|^\alpha) \|U^n(r, s)\|_\infty + \|U^n(s, t)\|_\infty \\ (34) \quad & + (Bd^3 M^2 \Gamma_R + 2d^3 M^2 C_1 \Gamma_R) |t - r|^{\alpha+\beta} \Delta_n^{2\alpha-\beta} \end{aligned}$$

Like the proof of Lemma 8, we divide the proof into two parts. We first prove that there exist a small enough constant $\delta > 0$ and a large enough constant $C_4(\delta)$, both independent of n , such that for $|t - r| < \delta$, $|U^n(r, t)| \leq C_4(\delta) |t - r|^{\alpha+\beta} \Delta_n^{2\alpha-\beta}$. And we prove it by induction. First we have $U_{t_k^n, t_k^n}^n = 0$ and $U_{t_k^n, t_{k+1}^n}^n = 0$. Suppose the bound holds for all pairs $r_0, t_0 \in D_n$ with $|t_0 - r_0| < |t - r|$. We pick $s \in D_n$ as the largest point between r and t such that $|s - r| \leq 1/2 |t - r|$. Then we also have $|(s + \Delta_n) - r| > 1/2 |t - r|$ and $|t - (s + \Delta_n)| < 1/2 |t - r|$.

$$\|U^n(r, t)\|_\infty \leq (1 + B|t - s|^\alpha) \|U^n(r, s)\|_\infty + \|U^n(s, t)\|_\infty + (Bd^3 M^2 \Gamma_R + 2d^3 M^2 C_1 \Gamma_R) |t - r|^{\alpha+\beta} \Delta_n^{2\alpha-\beta}$$

and

$$\begin{aligned} & \|U^n(s, t)\|_\infty \\ & \leq (1 + B\Delta_n^\alpha) \|U^n(s, s + \Delta_n)\|_\infty + \|U^n(s + \Delta_n, t)\|_\infty + (Bd^3 M^2 \Gamma_R + 2d^3 M^2 C_1 \Gamma_R) |t - s|^{\alpha+\beta} \Delta_n^{2\alpha-\beta} \\ & \leq \|U^n(s + \Delta_n, t)\|_\infty + (Bd^3 M^2 \Gamma_R + 2d^3 M^2 C_1 \Gamma_R) |t - r|^{\alpha+\beta} \Delta_n^{2\alpha-\beta} \end{aligned}$$

Therefore,

$$\begin{aligned} & \|U^n(r, t)\|_\infty \\ & \leq (1 + B\delta^\alpha) \|U^n(r, s)\|_\infty + \|U^n(s + \Delta_n, t)\|_\infty + 2(Bd^3 M^2 \Gamma_R + 2d^3 M^2 C_1 \Gamma_R) |t - r|^{\alpha+\beta} \Delta_n^{2\alpha-\beta} \\ & \leq \frac{2 + B\delta^\alpha}{2^{\alpha+\beta}} C_4(\delta) |t - r|^{\alpha+\beta} \Delta_n^{2\alpha-\beta} + 2(Bd^3 M^2 \Gamma_R + 2d^3 M^2 C_1 \Gamma_R) |t - r|^{\alpha+\beta} \Delta_n^{2\alpha-\beta} \end{aligned}$$

If we pick δ and $C_4(\delta)$ such that

$$B\delta^\alpha \leq 2^{\alpha+\beta} - 2$$

and

$$(1 - \frac{2 + B\delta^\alpha}{2^{\alpha+\beta}}) C_4(\delta) \geq 2(Bd^3 M^2 \Gamma_R + 2d^3 M^2 C_1 \Gamma_R),$$

Then $\|U^n(r, t)\|_\infty \leq C(\delta) |t - r|^{\alpha+\beta} \Delta_n^{2\alpha-\beta}$. We next extend the result to the case when $|t - r| > \delta$. We can always divide the interval $[r, t]$ into smaller intervals of length less than δ , specifically, for

n large enough, we consider $r = s_0 < s_1 < \dots < s_k = t$ where $s_i \in D_n$ and $1/2\delta < |s_i - s_{i-1}| < \delta$ for $i = 1, 2, \dots, k$. Then $k < 2|t - r|/\delta \leq 2/\delta$ and

$$\begin{aligned}
& \|U^n(r, t)\|_\infty \\
& \leq (1 + B|s_1 - s_0|^\alpha) \|U^n(s_0, s_0)\|_\infty + \|U^n(s_1, s_2)\|_\infty + (Bd^3 M^2 \Gamma_R + 2d^3 M^2 C_1 \Gamma_R) |t - r|^{\alpha+\beta} \Delta_n^{2\alpha-\beta} \\
& \leq \sum_{i=1}^k (1 + B\delta^\alpha) \|U^n(s_{i-1}, s_i)\|_\infty + k(Bd^3 M^2 \Gamma_R + 2d^3 M^2 C_1 \Gamma_R) |t - r|^{\alpha+\beta} \Delta_n^{2\alpha-\beta} \\
& \leq (1 + B\delta^\alpha) C_4(\delta) \Delta_n^{2\alpha-\beta} \sum_{i=1}^k |s_i - s_{i-1}|^{\alpha+\beta} + k(Bd^3 M^2 \Gamma_R + 2d^3 M^2 C_1 \Gamma_R) |t - r|^{\alpha+\beta} \Delta_n^{2\alpha-\beta} \\
& \leq (1 + B\delta^\alpha) C_4(\delta) |t - r|^{\alpha+\beta} \Delta_n^{2\alpha-\beta} + \frac{2}{\delta} (Bd^3 M^2 \Gamma_R + 2d^3 M^2 C_1 \Gamma_R) |t - r|^{\alpha+\beta} \Delta_n^{2\alpha-\beta} \\
& \leq C_4 |m - k|^{\alpha+\beta} \Delta_n^{2\alpha-\beta}
\end{aligned}$$

for $C_4 \geq (1 + B\delta^\alpha) C_4(\delta) + 2(Bd^3 M^2 \Gamma_R + 2d^3 M^2 C_1 \Gamma_R)/\delta$.

□

We are now ready to prove Proposition 3.

Proof of Proposition 3. From Lemma 10, we have

$$\|U^n(0, t)\|_\infty \leq C_4 t^{\alpha+\beta} \Delta_n^{2\alpha-\beta}.$$

Then

$$\begin{aligned}
|\hat{X}_i^n(t) - X_i^n(t)| & \leq |U_i^n(0, t)| + \sum_{j=1}^d \sum_{l=1}^d \sum_{m=1}^d |\partial_l \sigma_{i,j}(X(0)) \sigma_{l,m}(X(0))| |R_{m,j}^n(0, t)| \\
& \leq C_4 t^{\alpha+\beta} \Delta_n^{2\alpha-\beta} + d^3 M^2 \Gamma_R t^\beta \Delta_n^{2\alpha-\beta} \\
& \leq (C_4 + d^3 M^2 \Gamma_R) \Delta_n^{2\alpha-\beta}.
\end{aligned}$$

□

REFERENCES

- [1] C. Bayer, P. Friz, S. Riedel, and J. Schoenmakers. From rough path estimates to multilevel Monte Carlo. <http://arxiv.org/pdf/1305.5779v1.pdf>, 2013.
- [2] A. Beskos and G. Roberts. Exact simulation of diffusions. *Annals of Applied Probability*, 15: 2422–2444, 2005.
- [3] A. Beskos, O. Papaspiliopoulos, and G. Roberts. Retrospective exact simulation of diffusion sample paths with applications. *Bernoulli*, 12(6):1077–1098, 2006. ISSN 1350-7265. doi: 10.3150/bj/1165269151. URL <http://dx.doi.org/10.3150/bj/1165269151>.
- [4] A. Beskos, S. Peluchetti, and G. Roberts. ϵ -strong simulation of the Brownian path. *Bernoulli*, 18(4):1223–1248, 2012. ISSN 1350-7265. doi: 10.3150/11-BEJ383. URL <http://dx.doi.org/10.3150/11-BEJ383>.
- [5] J. Blanchet and X. Chen. Steady-state simulation for reflected Brownian motion and related networks. <http://arxiv.org/pdf/1202.2062.pdf>, 2013.
- [6] N. Chen and Z. Huang. Localization and exact simulation of brownian motion-driven stochastic differential equations. *Mathematics of Operations Research*, 38:591–616, 2013.
- [7] A.M. Davie. Differential equations driven by rough paths: An approach via discrete approximation. <http://arxiv.org/abs/0710.0772>.

- [8] P. Fritz and N. Victoir. *Multidimensional Stochastic Processes as Rough Paths: Theory and Applications*, volume 120. Cambridge University Press, 2010.
- [9] T.J. Lyons. Differential equations driven by rough signals. *Rev. Mat. Iberoamericana*, 14(2): 215–310, 1998.
- [10] M. Pollock, A. Johansen, and G. Roberts. On the exact and ε -strong simulation of (jump) diffusions. <http://arxiv.org/pdf/1302.6964v2.pdf>, 2014.
- [11] C. Rhee and P. Glynn. A new approach to unbiased estimation of sdes. <http://arxiv.org/pdf/1207.2452.pdf>, 2012.
- [12] J.M. Steele. *Stochastic Calculus and Financial Application*. Springer-Verlag, 2001.

COLUMBIA UNIVERSITY, DEPARTMENT OF INDUSTRIAL ENGINEERING & OPERATIONS RESEARCH, 340 S. W. MUDD BUILDING, 500 W. 120 STREET, NEW YORK, NY 10027, UNITED STATES.

E-mail address: jose.blanchet@columbia.edu